

CERTAIN WELL-FACTORED CATEGORIES

By

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To my mother and to the memory of my father

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A new kind of category, called a well-factored category, is defined. This is a generalization of the category of all modules over all rings, the category of all fibre bundles with fixed fibre, and the category of all topological acts.

Two structure theorems are proved for well-factored categories. One is an embedding into a "product" category and the other is an embedding into the category of all left actions in a multiplicative category without unit. These embeddings preserve the factorization of morphisms given in the definition of a well-factored category.

A sufficient condition for a category of left actions to be a variety, as defined by Herrlich, is given. As a corollary, it is noted that the category of all modules over all rings is a variety.

INTRODUCTION

Let R be a ring and ${}_R\mathbf{M}$ be the category of all left R -modules. The study of the category ${}_R\mathbf{M}$ is already well developed and familiar. The author set out to study the category of all modules over all rings, denoted $\mathbf{RM}$. Thus, each ${}_R\mathbf{M}$ is a subcategory of $\mathbf{RM}$ and morphisms in $\mathbf{RM}$ are semilinear transformations (f,g) ; i.e., f is a ring homomorphism, g is a group homomorphism, and $g(rm) = f(r)g(m)$, for each r and m . The author then noted that numerous categories such as the category of all topological acts, the category of all compact acts, the category of all fibre bundles with fixed fibre, the category of all monoids acting on sets, and the category of all bimodules over all pairs of rings possess properties similar to those of $\mathbf{RM}$. Most importantly, in all these categories, we may uniquely factor each morphism as the composition of two morphisms belonging to corresponding classes of morphisms. Roughly speaking, for $\mathbf{RM}$, we have "module" morphisms of the form $(1,g)$ and "ring" morphisms of the form $(f,1)$. These examples and the generalizing concept of a well-factored category are presented in Chapter I.

In Chapter II, we prove two structure theorems for well-factored categories. We show that a well-factored category may be embedded in a very nice way into the "product"

of three categories. Then we show that a well-factored category may be embedded in a very nice way as a subcategory of the category of all left actions in a multiplicative category without unit.

Herrlich introduced in [1] an axiomatically defined varietal category which generalized the "varieties" of Lawvere and Linton. We give in Chapter III a sufficient condition for the category of all left actions in a multiplicative category without unit to be a varietal category in the sense of Herrlich. A corollary to this result is that the category of all modules over all rings is a variety. We have a generalization of this condition and list some corollaries to the generalization but, due to the pressures of time, we omit the proof. We hope to publish the complete results later.

An important part of the proof for our condition given in Chapter III is that the forgetful functor $U: \text{Lact}(\underline{N}) \longrightarrow \text{Ens} \times \text{Ens}$ has a left adjoint. We were partially motivated by the construction of a left adjoint for U when $\underline{N}$ is Ens with the product functor and the usual associativity transformation. This construction was given in [2].

CHAPTER I

PRELIMINARIES

We will use the notation of [3] throughout, unless stated otherwise. Also, familiarity on the part of the reader with Chapters I, II, and IV of [3] will be assumed. In this chapter, we will state the definition of a well-factored category, give some examples of well-factored categories, and state some known results which are not assumed to be background knowledge for the reader.

(1.1). Remark. For the following definitions, let $\underline{C}$ be a category and let Γ_1 and Γ_2 be two "classes" of subcategories of $\underline{C}$ such that each object C in $\underline{C}$ is an object in exactly one member of Γ_1 and in exactly one member of Γ_2 . Let f be a morphism in $\underline{C}$. Then f is said to be a Γ_i -morphism if and only if f is a morphism in some category in Γ_i , $i=1,2$. Note then that the composition of Γ_i -morphisms is a Γ_i -morphism, $i=1,2$.

(1.2). Notation. Ab will denote the category of all abelian groups. $\underline{R}$ will denote the category of all rings which do not necessarily have an identity. $\underline{R}^1$ will denote the category of all rings having an identity and all identity-preserving ring homomorphisms. Ens is the category of all sets.

(1.3). Definition. Let $\underline{N} = \bigcup_{\underline{A} \in \Gamma_1} \text{Mor } \underline{A}$ and $\underline{M} = \bigcup_{\underline{A} \in \Gamma_2} \text{Mor } \underline{A}$ or

vice versa. Let f and g be in $\underline{N}$, say $f:A \rightarrow B$ and $g:A' \rightarrow B'$. We say f and g are grid connected by $\underline{M}$ -morphisms if and only if there exist sequences $S_1, \dots, S_n$ of $\underline{N}$ -morphisms such that each S_i is not the empty sequence, the elements of each S_i need not be composable, $S_1 = (f)$, $S_n = (g)$, and, for each $i = 1, \dots, n-1$, one of the following (or one of the following with S_i and S_{i+1} interchanged) occurs:

$$(1) \quad S_i = (g_{n_i}, \dots, g_{k+1}, g_k, g_{k-1}, \dots, g_1)$$

$$S_{i+1} = (g_{n_i}, \dots, g_{k+1}, \hat{g}_k, g_{k-1}, \dots, g_1)$$

where $\hat{g}_k$ indicates g_k has been deleted from the sequence.

Also, g_k is an identity such that if $\underline{N}$ is the collection of all Γ_i -morphisms, then $\text{cod } g_{k-1}$, $\text{dom } g_k$, and $\text{dom } g_{k+1}$ are all objects in the same Γ_j -subcategory of $\underline{C}$, $i=1,2$ and $i \neq j$. If $k = n_i$, then we have the above conditions for k and $k-1$. If $k = 1$, then we have the above conditions for k and $k+1$.

$$(2) \quad S_i = (g_{n_i}, \dots, g_{k+1}, g_k, g_{k-1}, \dots, g_1)$$

$$S_{i+1} = (g_{n_i}, \dots, g_{k+1}, f, f', g_{k-1}, \dots, g_1)$$

where $f \circ f'$ is defined and is equal to g_k .

$$(3) \quad S_i = (g_{n_i}, \dots, g_{k+1}, g_k, g_{k-1}, \dots, g_1)$$

$$S_{i+1} = (g_{n_i}, \dots, g_{k+1}, f, g_{k-1}, \dots, g_1)$$

and there exist $\underline{M}$ -morphisms m and m' so that $m' \circ f = g_{k+1} \circ m$; i.e., the following diagram commutes:

$$\begin{array}{ccc} & f & \\ E & \xrightarrow{\quad} & W \\ m \downarrow & & \downarrow m' \\ E' & \xrightarrow{g_k} & W' \end{array} .$$

(1.4). Definition. We say that $\underline{C}$ is well-factored by

Γ_1 and Γ_2 if and only if the following three conditions hold.

(1) If f_1 and f_2 are Γ_i -morphisms as indicated by subscript, then $f_1 \circ f_2$ is a Γ_i -morphism implies f_j is an identity, where $i \neq j$, $i = 1, 2$, and $j = 1, 2$.

(2) If f_1 is grid connected to g_1 by Γ_2 -morphisms and f_2 is grid connected to g_2 by Γ_1 -morphisms, where $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ and $A \xrightarrow{g_1} E \xrightarrow{g_2} B$, then $f_2 \circ f_1 = g_2 \circ g_1$.

(3) For each morphism f in $\underline{C}$, there is a unique factorization $f = h \circ g$ such that g is a Γ_1 -morphism and h is a Γ_2 -morphism.

We will denote this situation by $\underline{C} = \Gamma_1 \times \Gamma_2$.

(1.5). Example. Let S be a monoid. We may consider S as a category in the usual manner. If S is well-factored by Γ_1 and Γ_2 , then the cardinality of Γ_i is one, $i = 1, 2$. Also, S is isomorphic to $\Gamma_1 \times \Gamma_2$, where Γ_i is identified with its member, for $i = 1, 2$, and $\times$ denotes monoid product.

Proof. We must put off the proof that this example is true until the next chapter.

(1.6). Example. Let G be a group. Then G may be considered, in the usual manner, as a category. Suppose G is well factored by Γ_1 and Γ_2 . Then the cardinality of Γ_i is one, for $i = 1, 2$. Identifying Γ_i with its member, for $i = 1, 2$, we have $G = \Gamma_1 \times \Gamma_2$, where $\times$ denotes the direct product of groups.

Proof. We must put off the proof that this example is true until the next chapter.

(1.7). Definition. Let $\underline{C}_1, \underline{C}_2$, and $\underline{C}_3$ be categories. Define the category $\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3$ as follows: the object

class of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ is equal to the object class of $\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3$. Also, a morphism in $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ is of the form $(f, g): (C_1, C_2, C_3) \rightarrow (C_1', C_2', C_3')$, where $f: C_1 \rightarrow C_1'$ is a morphism in $\underline{C}_1$ and $g: C_2 \rightarrow C_2'$ is a morphism in $\underline{C}_2$. It is clear that $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ is indeed a category.

(1.8). **Proposition.** Let $\underline{C}_1, \underline{C}_2$, and $\underline{C}_3$ be categories. Let C_1 be an object in $\underline{C}_1$. Let $\Gamma_1(C_1)$ be the subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ given by:

$$\text{Ob } \Gamma_1(C_1) = \{(C_1, C_2, C_3) \mid C_1 \in \text{Ob } \underline{C}_1, 1 = 2, 3\}$$

and $\text{Mor } \Gamma_1(C_1) = \{(C_1, C_2, C_3) \xrightarrow{(1, g)} (C_1, C_2', C_3') \mid g: C_2 \rightarrow C_2' \text{ is a } \underline{C}_2\text{-morphism and } 1 = 1_{C_1}\}$. Let $\Gamma_1 = \{\Gamma_1(C_1) \mid C_1 \in \text{Ob } \underline{C}_1\}$. Define Γ_2 analogously; i.e., define Γ_2 by restricting the second coordinate. Then any subcategory $\underline{A}$ of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$, in which (3) of 1.4 holds and such that $(1, 1): (A_1, A_2, A_3) \rightarrow (A_1, A_2, A_3')$ is an $\underline{A}$ -morphism, implies $A_3 = A_3'$ is well-factored by the restrictions of Γ_1 and Γ_2 to $\underline{A}$. We call this the standard well-factorization of $\underline{A}$.

Proof. Let $\underline{A}$ be such a category. It is clear that each object in $\underline{A}$ is an object in exactly one member of Γ_1 and in exactly one member of Γ_2 . First, we prove that (1) of 1.4 holds in $\underline{A}$. Let $(1, \alpha): (B_1, B_2, B_3) \rightarrow (B_1, A_2, A_3)$ and $(\beta, 1): (C_1, B_2, C_3) \rightarrow (B_1, B_2, B_3)$. Then $(1, \alpha) \circ (\beta, 1) = (\gamma, 1)$ implies $\alpha = 1$. Then by hypothesis $B_3 = A_3$. Thus, $(1, \alpha)$ is an identity. Similarly, $(1, \alpha) \circ (\beta, 1) = (1, \gamma)$ implies $(\beta, 1)$ is an identity. Thus, (1) of 1.4 holds. Suppose

$$(A_1, A_2, A_3) \xrightarrow{(1, f_1)} (A_1, B_2, B_3) \xrightarrow{(f_2, 1)} (C_1, B_2, C_3) \text{ and}$$

$(A_1, A_2, A_3) \xrightarrow{(1, g_1)} (A_1, B_2, B'_3) \xrightarrow{(g_2, 1)} (C_1, B_2, C_3)$ where
 $(1, f_1)$ and $(1, g_1)$ are grid connected by Γ_2 -morphisms and
 $(f_2, 1)$ and $(g_2, 1)$ are grid connected by Γ_1 -morphisms. It
suffices to prove $f_1 = g_1$ and $f_2 = g_2$. We will prove $f_1 = g_1$
since the proof that $f_2 = g_2$ is analogous. We use the
sequences of Γ_1 -morphisms from 1.3, where $S_1 = (f_1)$ and
 $S_n = (g_1)$. We will prove the following inductively: if
 $S_i = (g_{n_i}, \dots, g_1)$, then $g_{n_i} = (1, h_{n_i}), \dots, g_1 = (1, h_1)$,
 $h_{n_i} \circ \dots \circ h_1$ is defined and if $S_{i+1} = (g_{n_{i+1}}, \dots, g_1)$, $g_{n_{i+1}} =$
 $(1, h_{n_{i+1}}), \dots, g_1 = (1, h_1)$, then $h_{n_{i+1}} \circ \dots \circ h_1 = h_{n_i} \circ \dots \circ h_1$,
for $i = 1, \dots, n-1$. Suppose $i = 1$. By 1.3, we have three
possible cases.

(1) $S_1 = ((1, f_1))$ and $S_2 = ((1, 1), (1, f_1))$ or $S_2 =$
 $((1, f_1), (1, 1))$. Then $\text{cod}(1, f_1)$ and $\text{dom}(1, 1)$ are in the
same Γ_2 -category implies $\text{cod}(1, f_1) = \text{dom}(1, 1) = \text{cod}(1, 1)$.
Thus, the inductive statement is true here for $i = 1$.

(2) $S_1 = ((1, f_1))$ and $S_2 = ((1, f), (1, f'))$, where
 $(1, f) \circ (1, f') = (1, f_1)$. Then the inductive statement for
 $i = 1$ is obviously true here.

(3) $S_1 = ((1, f_1))$ and $S_2 = ((1, h))$ where there exist
 Γ_2 -morphisms $(m, 1)$ and $(m', 1)$ such that $(m', 1) \circ (1, h) =$
 $(1, f_1) \circ (m, 1)$. Thus, $h = f_1$, so the inductive statement
for $i = 1$ is true here.

Going from S_i to S_{i+1} is quite similar to the above.
Thus, $f_1 = g_1$. Similarly, $f_2 = g_2$. Hence, $(f_2, 1) \circ (1, f_1) =$
 $(g_2, 1) \circ (1, g_1)$. Thus, $\underline{A}$ is well-factored.

(1.9). Example. We now exhibit a product category

such that condition (3) of 1.4 does not hold in all of its subcategories. Let $\underline{C}$ be the full subcategory of $\text{Ens} \times \text{Ens} \cong \overline{\text{Ens} \times \text{Ens} \times 0}$ whose objects are pairs (X, Y) of sets X and Y which have the same cardinality. Let $\underline{C}$ have the standard well-factorization. Let $\text{inc}: \{1\} \longrightarrow \{1, 2\}$ be the inclusion function. Then $(\text{inc}, \text{inc}): (\{1\}, \{1\}) \longrightarrow (\{1, 2\}, \{1, 2\})$ is a $\underline{C}$ -morphism. However, it is easy to see that this has no Γ_1 - Γ_2 factorization.

(1.10). Example. We now exhibit a category $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ which is not well-factored in the standard way. Let $\underline{C}_1 = \underline{C}_2 = \underline{C}_3 = \text{Ens}$. Then it is easy to see that the morphism $(1, 1): (\{1\}, \{1\}, \{1\}) \longrightarrow (\{1\}, \{1\}, \{1, 2\})$ has infinitely many Γ_1 - Γ_2 factorizations. Note also that the condition $(1, 1): (C_1, C_2, C_3) \longrightarrow (C_1, C_2, C_3^i)$ is a morphism implies $C_3 = C_3^i$ fails.

(1.11). Example. Let $\underline{A}$ be a category. Let Σ be the diagram scheme given by $1 \longrightarrow 2$. Then $[\Sigma, \underline{A}]$ is the category of all diagrams in $\underline{A}$ over Σ . For convenience of notation, we will denote $t: A_1 \longrightarrow A_2$ as (A_1, A_2, t) . Let $\underline{C}_1 = \underline{C}_2 = \underline{A}$ and let $\underline{C}_3$ be the discrete category such that $\text{Ob} \underline{C}_3 = \text{Mor} \underline{A}$. Then $[\Sigma, \underline{A}]$ is a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$. Suppose $(1, 1): (A_1, A_2, t) \longrightarrow (A_1, A_2, t')$. Then $1 \circ t = t' \circ 1$ so $t = t'$. Suppose $(f, g): (A_1, A_2, t) \longrightarrow (A_1^i, A_2^i, t')$. Then $(1, g): (A_1, A_2, t) \longrightarrow (A_1, A_2^i, t' \circ f)$ and $(f, 1): (A_1, A_2^i, t' \circ f) \longrightarrow (A_1^i, A_2^i, t')$ and $(f, g) = (f, 1) \circ (1, g)$. Furthermore, this is the unique Γ_1 - Γ_2 factorization of (f, g) . Thus, by 1.8, $[\Sigma, \underline{A}]$ is well-factored. Special cases of interest are:

(1) If $\underline{A} = \underline{R}$, then $[\Sigma, \underline{A}]$ is denoted $\underline{R} \underline{R}$ and is well-factored.

(2) If $\underline{A} = \underline{R}^1$, then $[\Sigma, \underline{A}]$ is denoted by $\underline{R}^1$.

(3) If $\underline{A} = \text{Top}$, the category of all topological spaces, then $[\Sigma, \underline{A}]$ is denoted as Bun , the category of all bundles.

(1.12). Example. Let $\underline{R}^M$ be the category of all modules over all rings. That is, an object in $\underline{R}^M$ is a triple (R, M, p) where R is a ring, M is an abelian group, and $p: R \times M \rightarrow M$ is a function so that (R, M, p) is a left R -module. A morphism in $\underline{R}^M$ taking (R, M, p) into (R', M', p') is a pair (f, g) so that f is a ring morphism taking R into R' , g is a group morphism taking M into M' , and, for each r in R and m in M , we have that $g(rm) = f(r)g(m)$. Let $\underline{R}^M$ be the subcategory of $\underline{R}^M$ consisting of all R -modules and so that a morphism (f, g) in $\underline{R}^M$ is a morphism in $\underline{R}^M$ if and only if $f = 1_R$. Whenever possible, (R, M, p) is denoted by (R, M) . Let $\underline{R}^M$ be the subcategory of $\underline{R}^M$ consisting of all modules having carrier equal to M . A morphism (f, g) in $\underline{R}^M$ is a morphism in $\underline{R}^M$ if and only if $g = 1_M$. Let Γ_1 be the collection of all $\underline{R}^M$ such that R is a ring. Let Γ_2 be the collection of all $\underline{R}^M$ such that M is an abelian group. Then $\underline{R}^M = \Gamma_1 \times \Gamma_2$.

Proof. Since Γ_1 and Γ_2 are the standard well-factorization, we may apply 1.8. Suppose $(1, 1): (R, M, p) \rightarrow (R, M, p')$. In the domain, $rm = p(r, m)$. In the codomain, $r * m = p'(r, m)$. Thus, $p'(r, m) = r * m = 1(r) * 1(m) = 1(rm) = 1(p(r, m)) = p(r, m)$. This says $p' = p$. Suppose $(f, g): (R, M, p) \rightarrow (R', M', p')$. Then $(1, g): (R, M, p) \rightarrow (R, M', p' \circ (f \times 1))$ and $(f, 1): (R, M', p' \circ (f \times 1)) \rightarrow (R', M', p')$ and $(f, 1) \circ (1, g) = (f, g)$. Moreover, this is the unique Γ_1 - Γ_2 factorization. Then by 1.8 we have that

$$\underline{R}^{\underline{M}} = \Gamma_1 \times \Gamma_2.$$

(1.13). Example. Let $\underline{R}^{\underline{M}}_{\underline{R}}$ be the category of all bi-modules over all rings. An object (R, M, S, p_1, p_2) in $\underline{R}^{\underline{M}}_{\underline{R}}$ is such that R and M are rings, M is an abelian group, $p_1: R \times M \rightarrow M$, $p_2: M \times S \rightarrow M$, (R, M, p_1) is a left R -module, (M, S, p_2) is a right S -module, and, for each r in R and m in M and s in S , we have $(rm)s = r(ms)$. A morphism in $\underline{R}^{\underline{M}}_{\underline{R}}$ taking (R, M, S) into (R', M', S') is a triple (f, g, h) so that $f: R \rightarrow R'$ is a ring morphism, $h: S \rightarrow S'$ is a ring morphism, and $g: M \rightarrow M'$ is a group morphism so that, for each r in R and m in M and s in S , $g(rm) = f(r)g(m)$ and $g(ms) = g(m)h(s)$. Whenever possible, we denote (R, M, S, p_1, p_2) by (R, M, S) .

We define Γ_1 -subcategories analogously to Γ_1 -subcategories in 1.12 by holding the rings constant. We define Γ_2 -subcategories analogously to Γ_2 -subcategories in 1.12 by holding the abelian group constant. Then $\underline{R}^{\underline{M}}_{\underline{R}} = \Gamma_1 \times \Gamma_2$.

Proof. This proof is analogous to the proof of 1.12.

We change notation from (R, M, S, p_1, p_2) to $((R, S), M, (p_1, p_2))$ and from (f, g, h) to $((f, h), g)$.

(1.14). Example. Let $\underline{R}^{\underline{M}}_{\underline{R}}$ be the full subcategory of $\underline{R}^{\underline{M}}_{\underline{R}}$ determined by the objects of the form (R, M, R) . Then the well-factorization of $\underline{R}^{\underline{M}}_{\underline{R}}$ given in 1.13 induces a well-factorization of $\underline{R}^{\underline{M}}_{\underline{R}}$.

Proof. This is a straightforward application of 1.8.

(1.15). Examples. We define the categories $\underline{R}^1 \underline{M}$, $\underline{R}^1 \underline{M}_{\underline{R}^1}$, and $\underline{R}^1 \underline{M}_{\underline{R}^1}$ analogously to respectively $\underline{R}^{\underline{M}}$, $\underline{R}^{\underline{M}}_{\underline{R}}$, $\underline{R}^{\underline{M}}_{\underline{R}}$ by replacing $\underline{R}$ with $\underline{R}^1$. We also obtain well-factorizations

of these categories in an analogous fashion. Finally, $\underline{M}_R$ and $\underline{M}_R 1$ are well-factored, where now we have right modules rather than left modules.

Proof. It is clear that these examples are valid.

(1.16). Example. Let $\underline{C}$ be the category of all topological acts. An object in $\underline{C}$ is a triple $(S, X, p: S \times X \rightarrow X)$ such that S is a Hausdorff topological semigroup, X is a Hausdorff space, p is continuous and gives the multiplication, and, s and t in S and x in X , $(st)x = s(tx)$. We denote (S, X, p) by (S, X) whenever possible. A morphism in $\underline{C}$ taking (S, X) into (S', X') is a pair (f, g) such that $f: S \rightarrow S'$ is continuous and preserves the semigroup multiplication, $g: X \rightarrow X'$ is continuous, and, for each s in S and x in X , $g(sx) = f(s)g(x)$.

$\underline{C}$ is a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ where $\underline{C}_1$ is the category of all topological semigroups, $\underline{C}_2$ is the category of all Hausdorff spaces, and $\underline{C}_3$ is the discrete category having as objects all $p: S \times X \rightarrow X$ so that p is continuous and S and X are Hausdorff spaces. Then $\underline{C}$ is well-factored by the standard well-factorization.

Proof. This is an easy application of 1.8.

(1.17). Example. Let $\text{Comp } \underline{C}$ be the category of all compact topological acts. That is, $\text{Comp } \underline{C}$ is the full subcategory of $\underline{C}$ determined by the objects (S, X) in $\underline{C}$ such that S and X are compact. Then $\text{Comp } \underline{C}$ is well-factored by the factorization induced from $\underline{C}$.

Proof. We note that the induced factorization is just

the standard well-factorization of $\text{Comp } \underline{C}$. With this fact, the proof is an easy application of 1.8.

(1.18). Example. If G is a topological group, a right G -space is a topological space X along with a map $X \times G \longrightarrow X$. The image of (x, s) is denoted xs . We require the following:

(1) For each x in X and s and s' in G , $x(ss') = (xs)s'$.

(2) For each x in X , $x1 = x$, where 1 is the identity in G .

A G -space is called effective provided $xs = x$ implies $s = 1$. Let $X^* = \{(x, xs) \in X \times X \mid x \text{ is in } X \text{ and } s \text{ is in } G\}$, where X is an effective G -space. There is a function $t: X^* \longrightarrow G$ defined by $xt(x, x') = x'$ and which is called the translation function. A G -space X is called principal provided X is an effective G -space with a continuous translation function. A principal G -bundle is a bundle (X, p, B) , where X is a principal G -space. A morphism $(u, f): (X, p, B) \longrightarrow (X', p', B')$ between two principal G -bundles is a principal morphism provided $u: X \longrightarrow X'$ is a G -module homomorphism and is continuous. Let PBun be the category of principal bundles and principal morphisms. Let PBun_B be the subcategory of PBun obtained by holding B in (X, p, B) constant and PBun^X be the subcategory obtained by holding X in (X, p, B) constant. Then morphisms have identities in the "constant" coordinate. Let Γ_1 be the "class" of all PBun^X and let Γ_2 be the "class" of all PBun_B . Then $\text{PBun} = \Gamma_1 \times \Gamma_2$.

Proof. Since Γ_1 and Γ_2 are the factorization induced on PBun by the well-factorization of Bun given in 1.11,

it suffices to prove that condition (3) of 1.4 holds.

Suppose $(u, f): (X, p, B) \longrightarrow (X', p', B')$. Then $(1, f): (X, p, B) \longrightarrow (X, f \circ p, B')$ and $(u, 1): (X, f \circ p, B') \longrightarrow (X', p', B')$ and $(X, f \circ p, B')$ is an object in PBun . The rest is clear.

(1.19). Example. For further details about the category of fibre bundles and about 1.18, the reader should consult [4]. Two elements x and x' in a G -space X are said to be G -equivalent if and only if there exists s in G such that $xs = x'$. This is an equivalence relation, so we may form the quotient space, denoted X/G .

Let $d = (X, p, B)$ be a principal G -bundle and let F be a left G -space. The equation $(x, y)s = (xs, s^{-1}y)$ defines a right G -space structure on $X \times F$. Let X_F denote the quotient space $(X \times F)/G$ and let $p_F: X_F \longrightarrow B$ be the factorization of $X \times F \xrightarrow{p \times 1} X \xrightarrow{p} B$ by the canonical map $X \times F \longrightarrow X_F$. Thus, for (x, y) in $X \times F$, $p_F((x, y)G) = p(x)$. Then (X_F, p_F, B) , denoted $d[F]$, is the fibre bundle over B with fibre F (viewed as a G -space) and associated principal bundle d . A fibre morphism from $d[F]$ to $d'[F]$ is a bundle morphism of the form $(u_F, f): d[F] \longrightarrow d'[F]$, where $(u, f): d \longrightarrow d'$ is a principal bundle morphism and u_F is obtained as follows: $u: X \longrightarrow X'$ induces a G -morphism $ux1: X \times F \longrightarrow X' \times F$ and u_F is the induced quotient map from $(X \times F)/G$ to $(X' \times F)/G$. Let $\text{Bun}[F]$ denote the category of all fibre bundles with fibre F . Let $\text{Bun}[F]^X$ be the subcategory obtained by holding X constant in d and $\text{Bun}[F]_B$ be the subcategory obtained by holding B constant in d . Let Γ_1 be the "class" of all $\text{Bun}[F]^X$ and let

Γ_2 be the "class" of all $\text{Bun}[F]_B$. Then $\text{Bun}[F] = \Gamma_1 \times \Gamma_2$.

Proof. As in 1.13, it suffices to prove (3) of 1.4.

Suppose $(u_F, f): (X \times F/G, p_F, E) \longrightarrow (X' \times F/G, p'_F, B')$. Then we have a unique $\Gamma_1 - \Gamma_2$ factorization

$$\begin{array}{ccc} (X_F, p_F, B) & \xrightarrow{(u_F, f)} & (X'_F, p'_F, B') \\ & \searrow (1, f) & \nearrow (u_F, 1) \\ & (X_F, p'_F \circ u_F, B') & \end{array}$$

as soon as we prove that $(X_F, p'_F \circ u_F, B')$ is a fibre bundle. However, this is a simple consequence of the fact that we are working with quotient maps and the details are left to the reader.

(1.20). Path Categories. Recall that a graph is a class of objects, a class of morphisms, a domain function, and a codomain function; i.e., a graph is a category without composition. Thus, there is an obvious forgetful functor from the category of all categories to the category of all graphs. (We should point out that a morphism of graphs consists of a pair of functions (F_1, F_2) , so that for graphs G_1 and G_2 , $F_1: \text{Ob } G_1 \longrightarrow \text{Ob } G_2$ and $F_2: \text{Mor } G_1 \longrightarrow \text{Mor } G_2$ so that if g is a G_1 -morphism, then $F_1(\text{dom } g) = \text{dom } F_2(g)$ and $F_1(\text{cod } g) = \text{cod } F_2(g)$.) This forgetful functor has a left adjoint; i.e., free categories (over graphs) exist. (These are also called path categories.) This construction is done by using finite sequences of "composable" arrows in the graph as morphisms in the free category and juxtaposition as composition. Given an object A in the graph, $()_A$ (the

empty sequence from A to A) is the identity morphism on A . The path category on a graph G is denoted $\text{Pa } G$. Two systems of notation will be used in path categories. We may denote morphisms pictorially as $A \xrightarrow{g_1} B \xrightarrow{g_2} C \xrightarrow{g_3} \dots \xrightarrow{g_n} D$ or as sequences as $(g_n, \dots, g_3, g_2, g_1)$. Note that when written as a sequence, the morphisms are written in their "composable" order. The identity morphism on A may be denoted as $A \xrightarrow{(\)} A$ or just by $(\)_A$. We will write $A \xrightarrow{f} B \xrightarrow{(\)} B \xrightarrow{g} C$ as (g, f) .

(1.21). Categorical Relations. Let $\underline{C}$ be a category and let E be a relation on $\text{Mor } \underline{C}$. Then E is called a categorical relation if and only if $f E g$ implies $\text{dom } f = \text{dom } g$ and $\text{cod } f = \text{cod } g$. Also, E is said to be compatible if and only if (1) $f E g$, $h \circ g$ defined, and $h \circ f$ defined implies $h \circ f E h \circ g$ and (2) $f E g$, $f \circ h$ defined, and $g \circ h$ defined implies $f \circ h E g \circ h$.

It is clear that $\bigcup_{A, B \in \text{Ob } \underline{C}} (\text{hom}(A, B) \times \text{hom}(B, A))$ is a categorical, compatible equivalence relation. Since the intersection of any family of categorical, compatible equivalence relations is a categorical, compatible equivalence relation, any subset of $\bigcup_{A, B \in \text{Ob } \underline{C}} (\text{hom}(A, B) \times \text{hom}(B, A))$ generates a categorical, compatible equivalence relation. Some sources refer to a categorical, compatible equivalence relation as a congruence relation.

Finally, if E is a categorical, compatible equivalence relation, then we may form the quotient category of $\underline{C}$ by E , denoted by $\underline{C}/E$, as follows: $\text{Ob } \underline{C}/E = \text{Ob } \underline{C}$ and $\text{Mor}(\underline{C}/E) =$

$(\text{Mor } \underline{\mathcal{C}})/E$, $\text{dom}[f] = \text{dom } f$ and $\text{cod}[f] = \text{cod } f$, and $[f] \circ [g]$
 $= f \circ g$, where we write a morphism in $\underline{\mathcal{C}}/E$ with representative
 f in $\underline{\mathcal{C}}$ as $[f]$.

CHAPTER II

TWO EMBEDDING THEOREMS

In this chapter we show the equivalence of the general concept of well-factored category to two special kinds of well-factored categories.

(2.1). Definition. A multiplicative category without unit is a triple $\underline{M} = (\underline{M}, \otimes, \alpha)$ such that:

(a) $\underline{M}$ is a category.

(b) $\otimes: \underline{M} \times \underline{M} \rightarrow \underline{M}$ is a bifunctor.

(c) For all objects L, M, N in $\underline{M}$, $\alpha_{L,M,N}: L \otimes (M \otimes N) \rightarrow (L \otimes M) \otimes N$ is an isomorphism and is natural in L, M , and N . We will usually write $\alpha_{L,M,N}$ as just α .

(d) For all objects L, M, N , and P of $\underline{M}$, the following diagram commutes:

$$\begin{array}{ccccc}
 & & (L \otimes M) \otimes (N \otimes P) & & \\
 & \swarrow \alpha & & \searrow \alpha & \\
 L \otimes (M \otimes (N \otimes P)) & & & & ((L \otimes M) \otimes N) \otimes P \\
 \downarrow 1 \otimes \alpha & & & & \uparrow \alpha \otimes 1 \\
 L \otimes ((M \otimes N) \otimes P) & \xrightarrow{\alpha} & & & (L \otimes (M \otimes N)) \otimes P
 \end{array}$$

(2.2). Remark. Definition 2.1 and the definitions of a semigroup and of an act are motivated by similar definitions in [5]. Conditions a, b, and c do not imply condition d. For example, let $\underline{M} = {}_R \underline{M}$, where R is a commutative

ring with identity. Let $\otimes$ be the usual tensor product of modules over R . Let α be defined by: $1 \otimes (m \otimes n) \mapsto (-1 \otimes m) \otimes n$. Then conditions a, b, and c hold but d fails, if $\text{char } R \neq 2$.

(2.3). Definition. A semigroup in $\underline{N}$ is a pair (S, u) such that the following hold:

(a) S is an object in $\underline{N}$ where $\underline{N} = (\underline{N}, \otimes, \alpha)$.

(b) $S \times S \xrightarrow{u} S$ is a morphism in $\underline{N}$.

(c) The following diagram commutes:

$$\begin{array}{ccc}
 S \otimes (S \otimes S) & \xrightarrow{1 \otimes u} & S \otimes S \\
 \alpha \downarrow & & \downarrow u \\
 (S \otimes S) \otimes S & & S \\
 u \otimes 1 \downarrow & \xrightarrow{u} & \\
 S \otimes S & & S
 \end{array}$$

Whenever possible, we denote (S, u) as S .

(2.4). Definition. If $\underline{N}$ is a multiplicative category without unit, then $\text{Sgp}(\underline{N})$ is the category of semigroups in $\underline{N}$. A morphism in $\text{Sgp}(\underline{N})$ from (S, u) to (T, v) is a morphism $f: S \rightarrow T$ in $\underline{N}$ so that the following diagram commutes:

$$\begin{array}{ccc}
 S \otimes S & \xrightarrow{u} & S \\
 f \otimes f \downarrow & & \downarrow f \\
 T \otimes T & \xrightarrow{v} & T
 \end{array}$$

Composition is induced from $\underline{N}$. It is clear that $\text{Sgp}(\underline{N})$ is indeed a category.

(2.5). Definition. Let (S, u) be a semigroup in $\underline{N} = (\underline{N}, \otimes, \alpha)$ and let M be an object in $\underline{N}$. Then a left action of (S, u) on M in $\underline{N}$ is a triple $((S, u), M, \text{SOM} \xrightarrow{z} M)$ where z

is a morphism in $\underline{M}$ and the following diagram commutes:

$$\begin{array}{ccc}
 S \otimes (S \otimes M) & \xrightarrow{1 \otimes z} & S \otimes M \\
 \alpha \downarrow & & \downarrow z \\
 (S \otimes S) \otimes M & & M \\
 u \otimes 1 \downarrow & & \\
 S \otimes M & \xrightarrow{z} & M
 \end{array}$$

Whenever possible, we will denote $((S, u), M, S \otimes M \xrightarrow{z} M)$ by (S, M, z) .

(2.6). Definition. $\text{Lact}(\underline{N})$ is the category of all left actions in $\underline{N}$. A morphism in $\text{Lact}(\underline{N})$ taking (S, M, z) into (T, N, z') is a pair (f, g) where $f: S \rightarrow T$ is a morphism in $\text{Sgp}(\underline{N})$ and $g: M \rightarrow N$ is a morphism in $\underline{M}$ ($\underline{N} = (\underline{M}, \otimes, \alpha)$) and the following diagram commutes:

$$\begin{array}{ccc}
 S \otimes M & \xrightarrow{z} & M \\
 f \otimes g \downarrow & & \downarrow g \\
 T \otimes N & \xrightarrow{z'} & N
 \end{array}$$

The composition is induced from $\underline{M}$. It is clear that $\text{Lact}(\underline{N})$ is indeed a category.

(2.7). Definition. Let $\underline{M} = (\underline{M}, \otimes, \alpha)$ be a multiplicative category without unit. $\underline{M}$ is said to be exact if and only if α is always an identity morphism.

(2.8). Proposition. Let $\underline{C}$ be a category. Then there exists $\underline{N}$, an exact multiplicative category without unit, such that $\underline{C}$ is isomorphic to a subcategory of $\text{Lact}(\underline{N})$.

Proof. Let $\underline{E}$ be the category of all endofunctors of $\underline{C}$.

Then $\underline{N} = (\underline{M}, \text{composition}, \text{equality})$ is an exact multiplicative category without unit since the interchange law says composition is a bifunctor taking $\underline{M} \times \underline{M}$ into $\underline{M}$. Let C be an object in $\underline{C}$. Then let $\bar{C}: \underline{C} \rightarrow \underline{C}$ be defined by $\bar{C}(A) = C$, for each object A in $\underline{C}$, and $\bar{C}(f) = 1_C$, for each morphism f in $\underline{C}$. Then $\bar{C}$ is an endofunctor of $\underline{C}$.

Let K be the identity functor on $\underline{C}$. Then it is easy to see that $(K, 1_K)$ is a semigroup in $\underline{N}$. Define $F: \underline{C} \rightarrow \text{Lact}(\underline{N})$ by $F(C) = ((K, 1_K), \bar{C}, 1_{\bar{C}})$. The following diagram commutes:

$$\begin{array}{ccc}
 K \circ (K \circ \bar{C}) & \xrightarrow{1 \circ 1_{\bar{C}}} & K \circ \bar{C} \\
 \downarrow 1 & & \downarrow 1_{\bar{C}} \\
 (K \circ K) \circ \bar{C} & & \\
 \downarrow 1 \circ 1_{\bar{C}} & & \downarrow 1_{\bar{C}} \\
 K \circ \bar{C} & \xrightarrow{1_{\bar{C}}} & \bar{C}
 \end{array}$$

This shows that $F(C)$ is an object in $\text{Lact}(\underline{N})$. If f is a morphism in $\underline{C}$, then $F(f) = (1_K, \bar{f})$ where $\bar{f} = \{t_A\}_{A \in \text{Ob } \underline{C}}$ and, for each object A in $\underline{C}$, $t_A = f$. It is clear that $\bar{f}$ is a morphism in $\underline{M}$ and that $(1_K, \bar{f})$ is a morphism in $\text{Lact}(\underline{N})$. Thus, $F: \underline{C} \rightarrow \text{Lact}(\underline{N})$ is clearly a functor and is one-to-one on objects and on morphisms.

Now we show that $\text{in}(F)$ is a subcategory of $\text{Lact}(\underline{N})$. Suppose $F(f) \circ F(g)$ is defined. Then $\text{cod } F(g) = (K, \bar{B}, 1_{\bar{B}})$ and $\text{dom } F(f) = (K, \bar{C}, 1_{\bar{C}})$ and $\text{cod } F(g) = \text{dom } F(f)$. Thus, $\bar{B} = \bar{C}$ so $B = C$. But then $\text{cod } g = B = C = \text{dom } f$ so $f \circ g$ is defined. Hence, $F(f) \circ F(g) = F(f \circ g)$ so $\text{in}(F)$ is closed under composition. Therefore, $\text{in}(F)$ is a subcategory of $\text{Lact}(\underline{N})$ so $\underline{C}$ is isomorphic to $\text{in}(F)$.

(2.9). Remark. Although proposition 2.8 is a very general embedding theorem, it seems to have very little practical use. We will prove later an embedding theorem of well-factored categories into categories of left actions so that the embedding preserves Γ_1 -factorizations. This is a much more useful embedding. However, we will first prove an embedding theorem of a somewhat different nature.

(2.10). Theorem. Let $\underline{C}$ be a well-factored category. Then there exist categories $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$ such that $\underline{C}$ is isomorphic to a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ which is well-factored by the standard well-factorization. Moreover, each Γ_1 -morphism in $\underline{C}$ maps to a Γ_1 -morphism in the subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$, for $i = 1, 2$.

(2.11). Remark. Before proving 2.10, we apply it to prove 1.5 and 1.6. Then several preliminary propositions leading to the proof of 2.10 will be given.

(2.12). Remark. Note that if $\underline{A}$ is a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ so that $\underline{A}$ is well-factored in the standard way by Γ_1 and Γ_2 , then by letting $\Gamma_1' = \Gamma_2$ and $\Gamma_2' = \Gamma_1$, we have that $\underline{A} = \Gamma_1' \times \Gamma_2'$, provided (3) of 1.4 holds. However, Γ_1' and Γ_2' are not the standard well-factorization. It is clear that by letting $\underline{C}_1' = \underline{C}_2$, $\underline{C}_2' = \underline{C}_1$, and $\underline{C}_3' = \underline{C}_3$ that $\underline{A}$ is then well-factored in the standard way by Γ_1' and Γ_2' as a subcategory of $\overline{\underline{C}_1' \times \underline{C}_2' \times \underline{C}_3'}$. By 2.10, given any category $\underline{A}$ with any well-factorization, we can find an isomorphic copy of $\underline{A}$ so that the isomorphic copy is then well-factored in the standard manner by the corresponding well-factorization.

(2.13). Proof of 1.5. Define $H:S \longrightarrow \Gamma_1 \times \Gamma_2$ by $H(s) = (s_1, s_2)$, where $s = s_2 \circ s_1$. (We are using subscript to indicate Γ_1 -morphism.) By 2.10, there exist categories $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$ such that we have an embedding $F:S \longrightarrow \underline{C}_1 \times \underline{C}_2 \times \underline{C}_3$ which preserves Γ_1 -morphisms. Thus, $F(s_2)$ is a Γ_2 -morphism and $F(s_1)$ is a Γ_1 -morphism; i.e., $F(s_2) = (a, 1)$ and $F(s_1) = (1, b)$, for some a and b . Then we have $F(s_2) \circ F(s_1) = (a, 1) \circ (1, b) = (a, b) = (1, b) \circ (a, 1) = F(s_1) \circ F(s_2)$. But F is one-to-one on morphisms, so $s_2 \circ s_1 = s_1 \circ s_2$. We know H is well-defined since the Γ_1 - Γ_2 factorization of a morphism is unique. Let s and s' be in S . Then $s \circ s' = (s_2 \circ s_1) \circ (s'_2 \circ s'_1) = s_2 \circ (s_1 \circ s'_1) \circ s'_2 = s_2 \circ (s'_1 \circ s_1) \circ s'_2 = (s_2 \circ s'_2) \circ (s_1 \circ s'_1)$. Thus, $H(s \circ s') = (s_1 \circ s'_1, s_2 \circ s'_2) = (s_1, s_2) \circ (s'_1, s'_2) = H(s) \circ H(s')$, so H is a functor and a monoid morphism. $H(s) = H(s')$ implies $s = s_2 \circ s_1 = s'$ so $s = s'$. Thus, H is one-to-one.

Let (s_1, s_2) be an element of $\Gamma_1 \times \Gamma_2$. Then $s_2 \circ s_1$ is in S and $H(s_2 \circ s_1) = (s_1, s_2)$. Thus, H is onto. Since Γ_1 and Γ_2 are subcategories of S , Γ_1 and Γ_2 are submonoids of S . Hence, H is an isomorphism and so S is isomorphic to $\Gamma_1 \times \Gamma_2$.

(2.14). Proof of 1.6. We will first prove that Γ_2 is a normal subgroup of G . Since Γ_2 is a nonempty subcategory of G , 1 is in Γ_2 and Γ_2 is closed under multiplication. Let g_2 be in Γ_2 . Then $g_2^{-1} = h_2 \circ h_1$, where h_2 and h_1 are unique elements of Γ_2 and Γ_1 respectively. Then $1 = g_2 \circ g_2^{-1} = g_2 \circ (h_2 \circ h_1) = (g_2 \circ h_2) \circ h_1 = 1 \circ 1$. By uniqueness of Γ_1 - Γ_2 factorization, $h_1 = 1$ so $g_2^{-1} = h_2$ and h_2 is in Γ_2 . Thus,

Γ_2 is a subgroup of G . Let g_1 be in Γ_1 and g_2 be in Γ_2 . Then by an argument similar to the one in 2.13, we have that $g_1 \circ g_2 = g_2 \circ g_1$. Thus, for each h in Γ_2 , $g \circ h \circ g^{-1} = g_2 \circ g_1 \circ h \circ g_1^{-1} \circ g_2^{-1} = g_2 \circ g_1 \circ g_1^{-1} \circ h \circ g_2^{-1} = g_2 \circ h \circ g_2^{-1}$ which is in Γ_2 . Thus, Γ_2 is a normal subgroup of G . Similarly, Γ_1 is a normal subgroup of G . Thus, by uniqueness of Γ_1 - Γ_2 factorization, $G \cong \Gamma_1 \times \Gamma_2$, where $\times$ now denotes direct product.

(2.15). Definition. Let $\underline{C}$ be a category which is well-factored by Γ_1 and Γ_2 . Then a new category $\underline{C}_1$ may be formed as follows. Form a graph denoted as G_1 by taking $\text{Ob } G_1 = \Gamma_1$. For each Γ_2 -morphism $g: A \rightarrow B$, there exist unique $\underline{A}$ and $\underline{B}$ in Γ_1 such that A is an object in $\underline{A}$ and B is an object in $\underline{B}$. Let there correspond a unique arrow $g: \underline{A} \rightarrow \underline{B}$ in the graph G_1 . Define γ on $\text{Pa}(G_1)$ by:

$$(1) (\underline{A} \xrightarrow{1_A} \underline{A}) \gamma (\underline{A} \xrightarrow{(\quad)} \underline{A}), \text{ for each object } A \text{ in } \underline{A}.$$

(2) $(\underline{W} \xrightarrow{f} \underline{V} \xrightarrow{g} \underline{U}) \gamma (\underline{W} \xrightarrow{g \circ f} \underline{U})$ whenever $g \circ f$ is defined.

(3) $(\underline{W} \xrightarrow{f} \underline{V}) \gamma (\underline{W} \xrightarrow{g} \underline{V})$ provided there exist Γ_1 -morphisms a and b so that the following diagram commutes:

$$\begin{array}{ccc} \underline{E} & \xrightarrow{f} & \underline{B} \\ \downarrow a & & \downarrow b \\ \underline{W} & \xrightarrow{g} & \underline{V} \end{array}.$$

Now γ is a categorical relation on $\text{Pa}(G_1)$. Let $\bar{\gamma}$ be the categorical, compatible equivalence relation generated by γ . Then let $\underline{C}_1 = (\text{Pa } G_1) / \bar{\gamma}$.

(2.16). Remark. Let $\underline{C}$ be a category which is well-factored by Γ_1 and Γ_2 . We may form a category $\underline{C}_2$ by interchanging 1 and 2 throughout definition 2.15.

(2.17). Proposition. Let γ be a categorical relation. Let $\bar{\gamma}$ be the categorical, compatible equivalence relation generated by γ . Then $\bar{\gamma}$ consists precisely of those pairs (g, g') of morphisms such that $\text{dom } g = \text{dom } g'$ and $\text{cod } g = \text{cod } g'$ and the following holds: there exist morphisms $g_1, \dots, g_n$ such that for each $i = 1, \dots, n$, $\text{dom } g_i = \text{dom } g$ and $\text{cod } g_i = \text{cod } g$ and, for each $i = 0, \dots, n$, (letting $g_0 = g$ and $g_{n+1} = g'$) there exist $g_i^{j,1}$ and $g_{i+1}^{j,1}$ such that $g_i = g_i^{k_1,1} \circ \dots \circ g_i^{1,1}$ and $g_{i+1} = g_{i+1}^{k_1,1} \circ \dots \circ g_{i+1}^{1,1}$ and, for each $j = 1, \dots, k_i$, $g_i^{j,1} \bar{E} g_{i+1}^{j,1}$ where E is $=$, γ , or γ^{-1} .

Proof. The proof of this proposition is routine and will be left to the reader.

(2.18). Proposition. Let h and h' be Γ_1 -morphisms in $\underline{C}$ where $\underline{C} = \Gamma_1 \times \Gamma_2$. Let $\text{Pa}(\underline{C}_2)$ be the free category formed in 2.16 and let h and h' denote the morphisms in $\text{Pa}(\underline{C}_2)$ corresponding respectively to h and h' . If $h \bar{\gamma} h'$, then in $\underline{C}$, h and h' are Γ_1 -morphisms grid connected by Γ_2 -morphisms.

Proof. In order to prove this proposition, it is necessary to make a careful analysis of what $h \bar{\gamma} h'$ means. Namely, we have morphisms $g_1, \dots, g_n$ as in 2.17. Note that we may avoid the possibility that some g_i is the empty morphism by, if so, putting an identity (corresponding to a Γ_1 -identity) alongside $g_1, \dots, g_n$ and a γ -related path

category identity alongside h and h' . Now we write down equivalent conditions in $\underline{C}$ to describe the process in 2.17. We have that there exist sequences of Γ_1 -morphisms $S_0, \dots, S_{n+1}$ such that $S_0 = (h)$, $S_{n+1} = (h')$, no S_i is empty and (since we may without loss of generality assume that $g_i^{j,i} = g_{i+1}^{j,i}$, for each j except some one value of j) for S_i and S_{i+1} we have (or with S_i and S_{i+1} interchanged) one of the following occurs:

(a) $1_A \times ()_{\underline{A}}$ means that A is an object in $\underline{A}$ so we have $S_i = (v_p, \dots, v_{k+1}, v_k, v_{k-1}, \dots, v_1)$ and

$S_{i+1} = (v_p, \dots, v_{k+1}, \hat{v}_k, v_{k-1}, \dots, v_1)$ where $\hat{v}_k$ means v_k has been deleted from the sequence, v_k is an identity, and $\text{cod } v_{k-1}$, $\text{dom } v_{k+1}$, and $\text{cod } v_k = \text{dom } v_k$ are all in the same Γ_2 -category.

(b) $S_i = (v_p, \dots, v_{k+1}, v_k, v_{k-1}, \dots, v_1)$ and

$S_{i+1} = (v_p, \dots, v_{k+1}, s, t, v_{k-1}, \dots, v_1)$ where $s \circ t = v_k$.

(c) $S_i = (v_p, \dots, v_{k+1}, v_k, v_{k-1}, \dots, v_1)$ and

$S_{i+1} = (v_p, \dots, v_{k+1}, w, v_{k-1}, \dots, v_1)$ where there exist Γ_2 -morphisms a and b so that $a \circ v_k = w \circ b$.

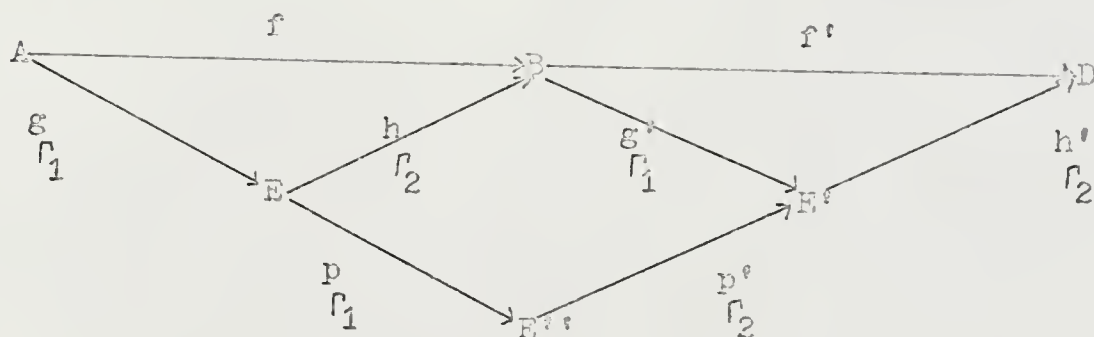
Then (a) corresponds to (1) in the definition of γ , (b) corresponds to (2) in the definition of γ , and (c) corresponds to (3) in the definition of γ . Thus, by 1.3, h and h' are Γ_1 -morphisms grid connected by Γ_2 -morphisms.

(2.19). Proposition. Let h and h' be Γ_2 -morphisms in $\underline{C}$ where $\underline{C} = \Gamma_1 \times \Gamma_2$. Let $\text{Pa}(G_1)$ be the path category formed in 2.15 and let h and h' denote the morphisms in $\text{Pa}(G_1)$ corresponding respectively to h and h' . If $h \gamma h'$,

then in $\underline{C}$, h and h' are Γ_2 -morphisms grid connected by Γ_1 -morphisms.

Proof. The proof is the same as for 2.18, with 1 and 2 interchanged.

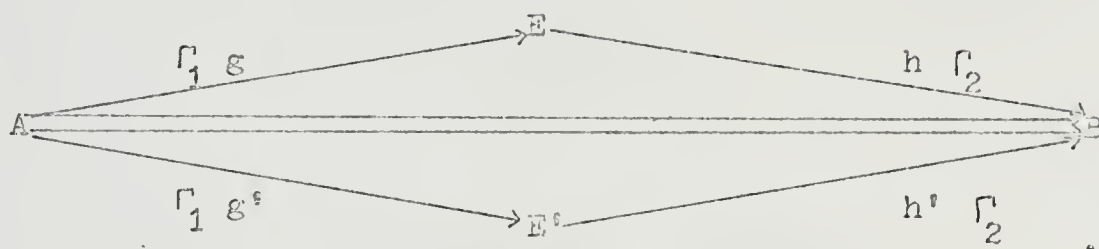
(2.20). Proof of 2.10. Let $\underline{C}_1$ and $\underline{C}_2$ be the categories formed in 2.15 and 2.16 respectively. Let $\underline{C}_3$ be the discrete category formed by the objects of $\underline{C}$. Let A be an object in $\underline{C}$. Then there exist unique A_1 in Γ_1 and unique A_2 in Γ_2 such that A is an object in A_1 and A is an object in A_2 . Define $F: \underline{C} \rightarrow \overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ on objects by $F(A) = (A_1, A_2, A)$. Suppose $f: A \rightarrow B$ is a morphism in $\underline{C}$. Then there exist a unique factorization $f = h \circ g$ where g is a Γ_1 -morphism and h is a Γ_2 -morphism. Then define $F(f) = ([h], [g]): (A_1, A_2, A) \rightarrow (B_1, B_2, B)$. It is clear that F is well-defined. Now we must show that F is a functor. Since $1_A = 1_A \circ 1_A$ and since 1_A is both a Γ_1 -morphism and a Γ_2 -morphism, $F(1_A) = ([1_A], [1_A]) = 1_{(A_1, A_2, A)} = 1_{F(A)}$. Suppose $A \xrightarrow{f} B \xrightarrow{f'} D$. Then we have Γ_1 - Γ_2 factorizations as indicated by the following commutative diagram:



By definition of γ in 2.15 and in 2.16, $h \circ p'$ in $\text{Pa}(\underline{C}_1)$ and $p \circ g'$ in $\text{Pa}(\underline{C}_2)$. Thus, $[h] = [p']$ and $[p] = [g']$.

$F(f') \circ F(f) = ([h'], [g']) \circ ([h], [g]) = ([h'] \circ [h], [g'] \circ [g]) = ([h'] \circ [p'], [p'] \circ [g]) = ([h' \circ p'], [p' \circ g]) = F(f' \circ f)$. Consequently, F is a functor.

It is clear that F is one-to-one on objects. Suppose $F(f) = F(f')$ where f and f' are morphisms in $\underline{C}$. Then $\text{dom } f = \text{dom } f'$ and $\text{cod } f = \text{cod } f'$. Also, f and f' have Γ_1 - Γ_2 factorizations as indicated by the following diagram where the top and bottom triangles are commutative:



Thus, $F(f) = ([h], [g])$ and $F(f') = ([h'], [g'])$ so $[h] = [h']$ and $[g] = [g']$; i.e., as $\text{Pa}(G_1)$ and $\text{Pa}(G_2)$ morphisms, $h \bar{g} h'$ and $g \bar{g} g'$. By 2.18 and 2.19, h and h' are Γ_2 -morphisms and g and g' are Γ_1 -morphisms. By definition of a well-factored category, we then have $f = h \circ g = h' \circ g' = f'$.

Thus, F is one-to-one on morphisms.

We next show that $\text{im}(F)$ is a subcategory of $\overline{C_1 \times C_2 \times C_3}$. It suffices to show $\text{im}(F)$ is closed under composition.

Suppose $F(f) \circ F(g)$ is defined. Then $\text{dom } F(f) = (A_1, A_2, A)$ and $\text{cod } F(g) = (B_1, B_2, B)$ so $A = B$. But by definition of F , $A = \text{dom } f$ and $B = \text{cod } g$ so $f \circ g$ is defined. Hence, $F(f) \circ F(g) = F(f \circ g)$ so $\text{im}(F)$ is closed under composition.

Since we have that $\underline{C}$ is isomorphic to $\text{im}(F)$, we now

show $\text{in}(F)$ is well-factored in the standard manner. It suffices to show that each Γ_1 -morphism in $\underline{C}$ maps to a Γ_1 -morphism in $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$, $i = 1, 2$. If f is a Γ_1 -morphism, then $f = 1 \circ f$ is the Γ_1 - Γ_2 factorization of f so $F(f) = ([f], [1])$ and $[1]$ is always an identity in $\underline{C}_2$. Hence, $F(f)$ is a Γ_1 -morphism. Similarly, we can show that F preserves Γ_2 -morphisms. This completes the proof.

(2.21). Remark. Now that we have gone through the construction of $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$, we may ask what $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$ are for some familiar categories.

Suppose S is a monoid well-factored by Γ_1 and Γ_2 . Then as noted earlier, discounting empty subcategories, $\text{card}(\Gamma_1) = \text{card}(\Gamma_2) = 1$ and, identifying Γ_1 with its member and Γ_2 with its member, $S \cong \Gamma_1 \times \Gamma_2$ as a monoid. But by 2.10, S is isomorphic to a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$. We ask, "What are $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$?" According to the construction, $\underline{C}_1 = (\text{Pa } G_1)/\sim$ where G_1 is the graph of Γ_2 -morphisms. Then G_1 is Γ_2 considered as a graph. But we identify 1_{Γ_2} with $()_{G_1}$ and (f, g) with $(f \circ g)$. Finally, we say $f \sim g$ if there are Γ_1 -morphisms a and b so that $b \circ f = g \circ a$. But as noted in 2.13, $g \circ a = a \circ g$, so $b \circ f = a \circ g$. We then have two Γ_1 - Γ_2 factorizations of an S -morphism. Hence, $\underline{C}_1 \cong \Gamma_2$. Similarly, $\underline{C}_2 \cong \Gamma_1$ and $\text{card}(\underline{C}_3) = \text{card}(\text{Ob } S) = 1$. It is easy to see that $F: S \rightarrow \overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ from 2.10 is onto.

Similarly, if G is a group and is well-factored by Γ_1 and Γ_2 , then $\text{card}(\Gamma_1) = \text{card}(\Gamma_2) = 1$ and, making the identifications of above, $\underline{C}_1 \cong \Gamma_2$, $\underline{C}_2 \cong \Gamma_1$, and $\text{card}(\underline{C}_3) = 1$.

Now we consider $\underline{R}\underline{M}$, the category of all modules over all rings. $\text{Ob}(G_1) = \{ \underline{R}\underline{M} \mid R \text{ is a ring} \}$. Arrows in G_1 are $\underline{R}\underline{M} \xrightarrow{(g,1)} \underline{R'}\underline{M}$ so that $(g,1): (R, M, \cdot) \longrightarrow (R', M, *)$ for some M . Suppose $(g,1): (R, M, \cdot) \longrightarrow (R', M, *)$. The following diagram commutes:

$$\begin{array}{ccc} (R, M, \cdot) & \xrightarrow{(g,1)} & (R', M, *) \\ (1,0) \downarrow & & \downarrow (1,0) \\ (R, M', \cdot') & \xrightarrow{(g,1)} & (R', M', *') \end{array}$$

Consequently, $(g,1_M) \bar{\sim} (g,1_{M'})$. Given a ring morphism $g: R \longrightarrow R'$, $(g,1): (R, R', \cdot) \longrightarrow (R', R', *)$ where $r \cdot r' = g(r) * r'$.

Also, given $\underline{R}\underline{M} \xrightarrow{(g,1)} \underline{R'}\underline{M} \xrightarrow{(g',1)} \underline{R''}\underline{M}$, we have $(g,1): (R, M, \cdot) \longrightarrow (R', M, *)$ and $(g',1): (R', M, *) \longrightarrow (R'', M, *')$.

The following diagram commutes where $r \cdot' m' = g(r) *' m'$:

$$\begin{array}{ccc} (R, M, \cdot) & \xrightarrow{(g,1)} & (R', M, *) \\ (1,0) \downarrow & & \downarrow (1,0) \\ (R, M', \cdot') & \xrightarrow{(g,1)} & (R', M', *') \end{array}$$

Thus, $((g',1_{M'}), (g,1_{M'})) \bar{\sim} (g' \circ g, 1_{M'})$, so $\underline{C}_1 \cong \underline{R}$.

G_2 has as objects all $\underline{R}\underline{M}$ such that M is an abelian group. Arrows in G_2 are $\underline{R}\underline{M} \xrightarrow{(1,g)} \underline{R}\underline{M'}$ so that for some R , $(R, M, \cdot) \xrightarrow{(1,g)} (R, M', *)$. Then the following diagram commutes:

$$\begin{array}{ccc} (0, M, \cdot) & \xrightarrow{(1,g)} & (0, M', \cdot) \\ (0,1) \downarrow & & \downarrow (0,1) \\ (R, M, \cdot) & \xrightarrow{(1,g)} & (R, M', *) \end{array}$$

Consequently, $(1_R, g) \bar{\sim} (1_0, g)$. Given any group morphism

$g: M \rightarrow M'$, where M and M' are abelian groups, $(1, g): (0, M, \cdot) \rightarrow (0, M', \cdot)$ so $(1, g): \underline{R}M \rightarrow \underline{R}M'$. Given $\underline{R}M \xrightarrow{(1, g)} \underline{R}M' \xrightarrow{(1, g')} \underline{R}M''$, we note that $((1_R, g'), (1_R, g)) \bar{=} (1_0, g' \circ g)$. Thus, we have composability of morphisms. Consequently, $\underline{C}_2 = (\text{Pa } \underline{G}_2) / \bar{=}$ is isomorphic to Ab . Our results then are that for $\underline{R}M$, $\underline{C}_1 \cong \underline{R}$, $\underline{C}_2 \cong \text{Ab}$, $\underline{C}_3$ is discrete, and $\text{card}(\underline{C}_3) = \text{card}(\underline{R}M)$.

(2.22) Proposition. Let $\underline{N} = (\underline{M}, \otimes, \alpha)$ be a multiplicative category without unit. Let (S, u) be an object in $\text{Sgp}(\underline{N})$, A be an object in $\underline{M}$, $((T, v), B, z)$ be an object in $\text{Lact}(\underline{N})$. Suppose $f: (S, u) \rightarrow (T, v)$ is a morphism in $\text{Sgp}(\underline{N})$ and that $g: A \rightarrow B$ is a morphism in $\underline{M}$. Then $((S, u), B, z \circ (f \otimes 1))$ is an object in $\text{Lact}(\underline{N})$.

Proof. This is equivalent to showing that the following diagram commutes:

$$\begin{array}{ccccc}
 S \otimes (S \otimes B) & \xrightarrow{1 \otimes (f \otimes 1)} & S \otimes (T \otimes B) & \xrightarrow{1 \otimes z} & S \otimes B \\
 \downarrow \alpha & & & & \downarrow f \otimes 1 \\
 (S \otimes S) \otimes B & & & & T \otimes B \\
 \downarrow u \otimes 1 & & & & \downarrow z \\
 S \otimes B & \xrightarrow{f \otimes 1} & T \otimes B & \xrightarrow{z} & B
 \end{array}$$

The proof that the above diagram commutes is routine and is left to the reader. We use that f is a semigroup morphism, α is natural, $\otimes$ is a bifunctor, and (T, B, z) is an object in $\text{Lact}(\underline{N})$.

(2.23). Proposition. Let $\underline{N}$ be as in 2.22. Let $\underline{C}_1 = \text{Sgp}(\underline{N})$ and $\underline{C}_2 = \underline{M}$. Let $\underline{C}_3$ be the discrete category with objects $z: S \otimes A \rightarrow A$ so that z is a morphism in $\underline{M}$. Then $\text{Lact}(\underline{N})$ is well-factored in the standard manner as a subcategory

of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$. We will refer to this as the standard well-factorization of $\text{Lact}(\underline{N})$.

Proof. By 1.8, it suffices to show that (3) of 1.4 holds and that if $(1,1):(S,A,z) \rightarrow (S,A,z')$ is a morphism in $\text{Lact}(\underline{N})$, then $z = z'$. Given the morphism $(1,1)$, the following diagram commutes:

$$\begin{array}{ccc}
 S \otimes A & \xrightarrow{z} & A \\
 \downarrow 1 \otimes 1 & & \downarrow 1 \\
 S \otimes A & \xrightarrow{z'} & A
 \end{array}$$

Thus, $z = z'$. Suppose $(f,g):((S,u),A,z_A) \rightarrow ((T,v),B,z_B)$ is a morphism in $\text{Lact}(\underline{N})$. By 2.22, $((S,u),B,z_B \circ (f \otimes 1))$ is an object in $\text{Lact}(\underline{N})$. Also, $g \circ z_A = z_B \circ (f \otimes g)$, since (f,g) is a morphism in $\text{Lact}(\underline{N})$. Hence, $g \circ z_A = z_B \circ (f \otimes 1) \circ (1 \otimes g)$ so $(1,g):(S,A,z_A) \rightarrow (S,B,z_B \circ (f \otimes 1))$ is a morphism in $\text{Lact}(\underline{N})$. It is clear that $(f,1):(S,B,z_B \circ (f \otimes 1)) \rightarrow (T,B,z_B)$ is a morphism in $\text{Lact}(\underline{N})$. Also, $(1,g)$ is a Γ_1 -morphism and $(f,1)$ is a Γ_2 -morphism and $(f,g) = (f,1) \circ (1,g)$. To see that the factorization is unique, it suffices to prove $(S,B,z_B \circ (f \otimes 1))$ is unique. If $(f,g) = (f',1) \circ (1,g')$, then $f' = f$, $g' = g$, and $\text{dom}(f',1) = (S,B,p)$. Also, $1 \circ p = z_B \circ (f \otimes 1)$ so $p = z_B \circ (f \otimes 1)$. Thus, the Γ_1 - Γ_2 factorization is unique so $\text{Lact}(\underline{N})$ is well-factored in the standard manner as a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$.

(2.24). Theorem. Let $\underline{C}$ be a well-factored in the standard manner subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$. Then there exists an exact multiplicative category $\underline{N} = (\underline{N}, \otimes, \alpha)$ without unit

such that $\underline{C}$ is isomorphic to a well-factored subcategory of $\text{Lact}(\underline{N})$ so that the well-factorization of $\text{Lact}(\underline{N})$ induces the well-factorization of the subcategory. Also, each Γ_i -morphism of $\underline{C}$ maps to a Γ_i -morphism in $\text{Lact}(\underline{N})$, $i = 1, 2$.

(2.25). Remark. Before proving 2.24, we obtain a preliminary result. In fact, this now summarizes 2.10 and 2.24.

(2.26). Theorem. Let $\underline{C}$ be a category and let Γ_1 and Γ_2 be "classes" of subcategories of $\underline{C}$. Then the following are equivalent:

(1) $\underline{C}$ is well-factored by Γ_1 and Γ_2 .

(2) There exist categories $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$ such that $\underline{C}$ is isomorphic to a subcategory of $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$ which is well-factored in the standard manner and such that each Γ_i -morphism in $\underline{C}$ maps to a Γ_i -morphism in $\overline{\underline{C}_1 \times \underline{C}_2 \times \underline{C}_3}$.

(3) There exists an exact multiplicative category $\underline{N} = (\underline{N}, \otimes, \alpha)$ without unit such that $\underline{C}$ is isomorphic to a subcategory of $\text{Lact}(\underline{N})$ well-factored by the factorization induced by the well-factorization of $\text{Lact}(\underline{N})$ and such that each Γ_i -morphism in $\underline{C}$ maps to a Γ_i -morphism in $\text{Lact}(\underline{N})$, for $i = 1, 2$.

Proof. It follows from 2.10 that (1) implies (2). It follows from 2.24 that (2) implies (3). It is clear that (3) implies (1).

(2.27). Proof of 2.24. Disjointify the categories $\underline{C}_1$ and $\underline{C}_2$. Form a graph Γ as follows: objects in Γ consist of pairs (A_1, A_2) where A_i is an object in $\underline{C}_i$, for $i = 1, 2$, or objects in Γ may themselves be objects in $\underline{C}_2$. For each object (A_1, A_2, A_3) in $\underline{C}$, we let there correspond a unique

morphism from (A_1, A_2) to A_2 which we will denote by A_3 .

Now we form another graph G as follows: Objects in G are n -tuples $(U_1, \dots, U_n)$ where each U_i is an object in $\underline{C}_1$ or $\underline{C}_2$ and n is greater than or equal one. A morphism f from $(U_1, \dots, U_n)$ to $(V_1, \dots, V_m)$ in G is a set of morphisms (in $\underline{C}_1$, $\underline{C}_2$, or Γ) denoted f_i or $f_{i,i+1}$ where $f_i: U_i \rightarrow V_{j(i)}$ is a morphism in $\underline{C}_1$ or in $\underline{C}_2$ and $f_{i,i+1}: (U_i, U_{i+1}) \rightarrow V_{j(i)}$ is a morphism in Γ . Also, we require $n \geq m$ and that, for each $i = 1, \dots, n$, i appears exactly once as a subscript of some f_i or $f_{i,i+1}$ where $f_{i,i+1}$ is counted as having two subscripts.

Given an object $(X_1, \dots, X_q)$ in $\text{Pa}(G)$, denote $\{1_{X_i}\}_{i=1}^q: (X_1, \dots, X_q) \rightarrow (X_1, \dots, X_q)$ as $1_{(X_1, \dots, X_q)}$. Suppose f and g are two morphisms in $\text{Pa}(G)$ going from $(U_1, \dots, U_n)$ to $(V_1, \dots, V_m)$. We obtain from f a new morphism $F = (F^t, \dots, F^1)$ and from g a new morphism $G = (G^s, \dots, G^1)$. Suppose $f = (f^k, \dots, f^1)$. Then let $F = (1_{(U_1, \dots, U_n)})$. Otherwise, $f = (f^k, \dots, f^1)$. Insert before f^1 the morphism $1_{(U_1, \dots, U_n)}$ and after f^k the morphism $1_{(V_1, \dots, V_m)}$. For $i = 1, \dots, k-1$, let $\text{cod } f^i = (U_1^i, \dots, U_{n(i)}^i)$. Insert between f^i and f^{i+1} the morphism $1_{(U_1^i, \dots, U_{n(i)}^i)}$. Calling this new sequence $(F^t, \dots, F^1)$, we see it is a path in $\text{Pa}(G)$ and we let $F = (F^t, \dots, F^1)$. G is obtained from g in a similar manner. Let i_0 be such that $1 \leq i_0 \leq n$. Then $F_{i_0}^1: U_{i_0} \rightarrow U_{i_0}$; in fact, $F_{i_0}^1 = 1_{U_{i_0}}$. Let $F_{i_0}^{s1} = F_{i_0}^1$. Exactly one of the following possibilities occurs:

$F_{i_0}^2: U_{i_0} \rightarrow W_p$, $F_{i_0, i_0+1}^2: (U_{i_0}, U_{i_0+1}) \rightarrow W_p$, or, finally,

$F_{i_0-1, i_0}^2: (U_{i_0-1}, U_{i_0}) \longrightarrow W_p$. Whichever of these occurs, we call it $F_{i_0}^2$. $F_p^3: W_p \longrightarrow W_p$ and $F_p^3 = 1_{W_p}$. Let $F_p^3 = F_{i_0}^3$. There are three possibilities for W_p which are analogous to the three possibilities for U_{i_0} . Whichever of these occurs we call $F_{i_0}^4$. Continuing in this manner, we obtain the sequence $(F_{i_0}^1, \dots, F_{i_0}^t)$. For $e = 1, \dots, t-1$, either $F_{i_0}^e \circ F_{i_0}^{e+1}$ is defined in C_1 or in C_2 or one of $F_{i_0}^e$ and $F_{i_0}^{e+1}$ is a Γ -morphism. By composing whatever adjacent morphisms are composable, we obtain a new sequence $(F_{i_0}^{T_{i_0}}, \dots, F_{i_0}^1)$ so that for $1 \leq e \leq T_{i_0}-1$, exactly one of $F_{i_0}^e$ and $F_{i_0}^{e+1}$ is a Γ -morphism. Similarly, from G we obtain $(G_{i_0}^{S_{i_0}}, \dots, G_{i_0}^1)$. If, for $i_0 = 1, \dots, n$, $(F_{i_0}^{T_{i_0}}, \dots, F_{i_0}^1) = (G_{i_0}^{S_{i_0}}, \dots, G_{i_0}^1)$, then we say γ . It is easy and is left to the reader to see that γ is a categorical, compatible equivalence relation. Let $\underline{A} = (\text{Pa } G)/\gamma$.

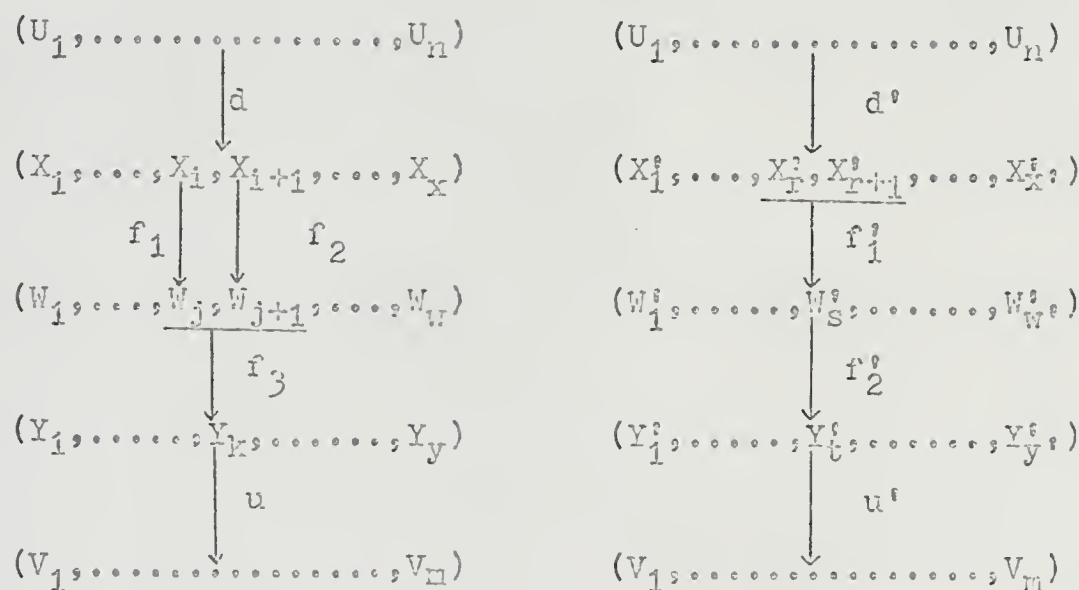
Let $\underline{M} = \underline{A}/\bar{\rho}$ where $\bar{\rho}$ is the categorical, compatible equivalence relation generated by ρ . The categorical relation ρ consists of all pairs (a, a') such that $a, a': (U_1, \dots, U_n) \longrightarrow (V_1, \dots, V_n)$ and there exist representatives b and b' in $\text{Mor Pa } G$ so that $a = b$ and $a' = b'$ and either $a = a'$ or at least one of the following two conditions holds or with a and a' interchanged at least one of the following two conditions holds:

- (1) $b = u \circ b_2 \circ b_1 \circ d$, $b_1, b_2 \in \text{Mor } G$, $f_1, f_2 \in b_1$, $f_1: X_1 \longrightarrow W_j$, $f_2: X_{j+1} \longrightarrow W_{j+1}$, $f_3 \in b_2$, and $f_3: (W_j, W_{j+1}) \longrightarrow X_k$. $b' = u' \circ b_2' \circ b_1' \circ d'$, $b_1', b_2' \in \text{Mor } G$, $f_1' \in b_1'$, $f_1': (X_1', X_{j+1}') \longrightarrow W_j'$, $f_2' \in b_2'$, $f_2': W_j' \longrightarrow W_{j+1}'$, so that $(X_1, X_{j+1}) = (X_1', X_{j+1}')$ and $Y_k = Y_k'$.

(2) $b = u \circ b_2 \circ b_1 \circ d$, $b_1, b_2 \in \text{Mor } G$, $f_1, f_2, f_3 \in b_1$, $f_1: X_{i-1} \rightarrow W_j$, $f_2: X_i \rightarrow W_j$, $f_3: X_{i+1} \rightarrow W_{j+1}$, f_1, f_2, f_3 are identities, $f_4 \in b_2$, and $f_4: (W_j, W_{j+1}) \rightarrow Y_k$. $b' = u' \circ b'_2 \circ b'_1 \circ d'$, $f'_1, f'_2 \in b'_1$, $f'_1: X_{i-1}^i \rightarrow W_S^i$ is an identity, $f'_2: (X_R^i, X_{R+1}^i) \rightarrow W_{S+1}^i$, and $f'_3 \in b'_2$ so that $f'_3: (W_S^i, W_{S+1}^i) \rightarrow Y_t^i$. Also, $(X_{i-1}, X_i, X_{i+1}) = (X_{i-1}^i, X_R^i, X_{R+1}^i)$ and $Y_k = Y_t^i$.

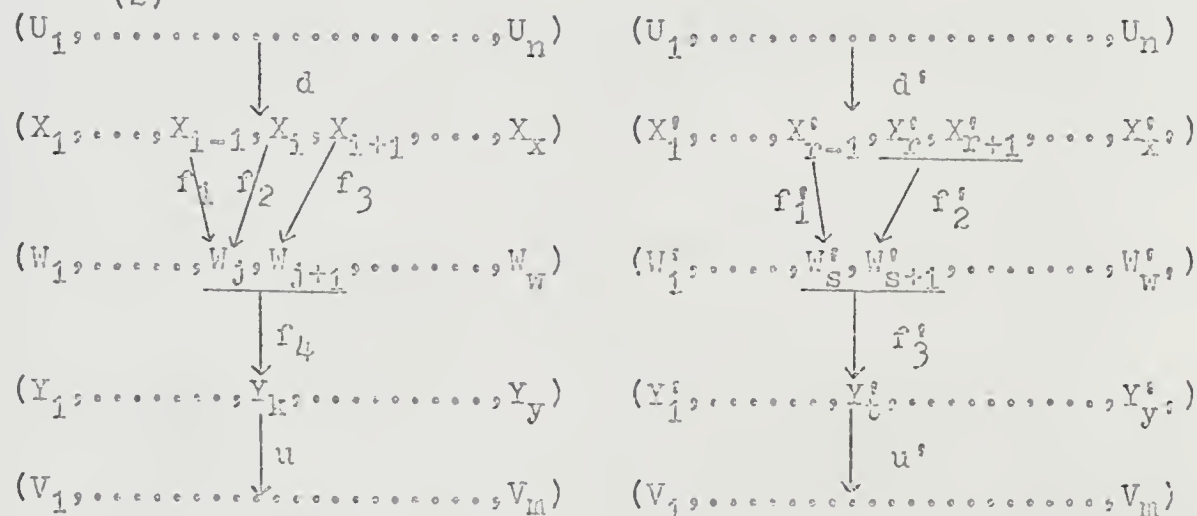
Pictorially, conditions (1) and (2) are as follows:

(1)



where $(X_i, X_{i+1}) = (X_R^i, X_{R+1}^i)$ and $Y_k = Y_t^i$.

(2)



where f_1, f_2, f_3 , and f'_1 are identities, $(X_{i-1}, X_i, X_{i+1}) = (X_{R-1}^i, X_R^i, X_{R+1}^i)$, and $Y_k = Y_t^i$.

Let $(U_1, \dots, U_n)$ and $(V_1, \dots, V_m)$ be objects in G .

Let $(U_1, \dots, U_n) \otimes (V_1, \dots, V_m) = (U_1, \dots, U_n, V_1, \dots, V_m)$; i.e., let it be $(W_1, \dots, W_{n+m})$ where, for $i = 1, \dots, n$, $W_i = U_i$ and, for $i = n+1, \dots, n+m$, $W_i = V_{i-n}$. Similarly, if $f: (U_1, \dots, U_n) \rightarrow (V_1, \dots, V_m)$, $f': (U'_1, \dots, U'_n) \rightarrow (V'_1, \dots, V'_m)$, $f = \{f_i\} \cup \{f_{i,i+1}\}$, and $f' = \{f'_j\} \cup \{f'_{j,j+1}\}$, then $f \otimes f': (U_1, \dots, U_n) \otimes (U'_1, \dots, U'_n) \rightarrow (V_1, \dots, V_m) \otimes (V'_1, \dots, V'_m)$ where we let $f \otimes f' = \{f_i\} \cup \{f_{i,i+1}\} \cup \{f'_j\} \cup \{f'_{j,j+1}\}$; i.e., $f \otimes f' = \{g_i\} \cup \{g_{i,i+1}\}$ where, for $i = 1, \dots, n$, $g_i = f_i$ and $g_{i,i+1} = f_{i,i+1}$ and, for $i = n+1, \dots, m$, $g_i = f'_{i-n}$ and $g_{i,i+1} = f'_{i-n,i-(n-1)}$.

Now we define $\otimes: \underline{M} \times \underline{M} \rightarrow \underline{M}$. On objects, $\otimes$ is the same as for G . Given $[[f]]$ and $[[g]]$, then $f = (f^k, \dots, f^1)$ and $g = (g^p, \dots, g^1)$. Assume that $k \geq p$. By definition of γ , we see that if $(X_1, \dots, X_n)$ is cod g , then $g\gamma(1, \dots, 1, g^p, \dots, g^1)$ where we take $k-p+1$ $(X_1, \dots, X_q)^i$'s and where $(X_1, \dots, X_q) = \text{cod } g^p = \text{cod } g$. We define $[[f]] \otimes [[g]] = [[h]]$ where $h = (f^k \otimes 1, \dots, f^{p+1} \otimes 1, f^p \otimes g^p, \dots, f^1 \otimes g^1)$. If $p > k$, then we lengthen f similarly.

For notational convenience, we will write $a \text{ (1) } b$ to indicate that a and b satisfy condition (1) in the definition of $\mathcal{R}$. We use a similar notation for condition (2). We need to show $\otimes$ is well-defined on morphisms. Suppose $[[v]] = [[v']]$ and $[[u]] = [[u']]$. We need to show $[[v]] \otimes [[u]] = [[v']] \otimes [[u']]$. Thus, it suffices to show $[[v]] \otimes [[u]] = [[v']] \otimes [[u]]$ and $[[v']] \otimes [[u]] = [[v']] \otimes [[u']]$. Since the proofs are similar, we show only that $[[v]] \otimes [[u]] = [[v']] \otimes [[u]]$. In $\underline{A}$, we have $[v] \mathcal{R} [v']$. Since $\mathcal{R}$ is a categorical, compatible, reflexive, and symmetric relation, $[v] \mathcal{R} [v']$ if and

only if there exist $v_1, \dots, v_n$ so that $[v] \mathcal{R} [v_1] \mathcal{R} [v_2] \mathcal{R} \dots \mathcal{R} [v']$. By the definition of $\mathcal{R}$, $[v] \mathcal{R} [v_1]$ means that there exist representatives e and e_1 so that $[e] = [v]$ and $[e_1] = [v_1]$ and condition (1) or (2) in the definition of $\mathcal{R}$ holds between e and e_1 or that $[v] = [v_1]$. However, we may assume without loss of generality that $e = v$ and $e_1 = v_1$. However, for $[v_1] \mathcal{R} [v_2]$, since we have already used v_1 in conjunction with v , we must use a new representative for v_1 , say $\bar{v}_1$. Thus, we have $v =$ or (1) or (2) $v_1 \times \bar{v}_1 =$ or (1) or (2) $v_2 \times \bar{v}_2 \dots \bar{v}_n =$ or (1) or (2) v' . Thus, it suffices to show: $b =$ or (1) or (2) w implies $[[b]] \otimes [[u]] = [[w]] \otimes [[u]]$ and that $b \times w$ implies $[[b]] \otimes [[u]] = [[w]] \otimes [[u]]$. To prove this, suppose b (1) or (2) w . Then $b = (b^k, \dots, b^1)$ and $w = (w^p, \dots, w^1)$ and $u = (u^m, \dots, u^1)$. Assume $k \geq m$ and $p \geq m$. Then $[[b]] \otimes [[u]] = [[(b^k \ 1, \dots, b^{m+1} \ 1, b^m \ u^m, \dots, b^1 \ u^1)]]$ and $[[w]] \otimes [[u]] = [[(w^p \ 1, \dots, w^{m+1} \ 1, w^m \ u^m, \dots, w^1 \ u^1)]]$. But then (1) or (2) will hold between the representatives given above for $[[b]] \otimes [[u]]$ and $[[w]] \otimes [[u]]$ so $[[b]] \otimes [[u]] = [[w]] \otimes [[u]]$. The other possibilities for k and p with m lead to differences only in notation. Now suppose $b \times w$. Then, recalling the definition of $\times$, we see that composition of morphisms of the form 1_{U_1} with either b or w preserves the relation $\times$, so $[[b]] \otimes [[u]] = [[w]] \otimes [[u]]$. Thus, $\otimes$ is well defined. It is easy to see that $\otimes$ is in fact a functor.

Letting α be equality, $\underline{M} = (\underline{M}, \otimes, \alpha)$ is clearly an exact multiplicative category without unit. Define $F: \underline{C} \rightarrow \text{Lact}(\underline{M})$ as follows. If (A_1, A_2, A_3) is an object in $\underline{C}$, let $F(A_1, A_2, A_3)$

$$= ((A_1, (A_1, A_1) \xrightarrow{[[1, 1]]} A_1), A_2, (A_1, A_2) \xrightarrow{[[A_3]]} A_3).$$

Note that we now identify A_i with (A_i) , $i = 1, 2$. Also, we identify f_i with $(\hat{f}_i)$ where f_i is a $\underline{C}_i$ -morphism, $i = 1, 2$. We do this since this identification embeds $\underline{C}_i$ into $\underline{M}$, $i = 1, 2$. Given $(f_1, f_2): (A_1, A_2, A_3) \longrightarrow (A_1^0, A_2^0, A_3^0)$ in $\underline{C}$, define $F(f_1, f_2) = ([f_1], [f_2])$. That $(A_1, (A_1, A_1) \xrightarrow{[[1, 1]]} A_1)$ is an object in $\text{Sgp}(\underline{N})$ follows immediately from the definition of γ . That $F(A_1, A_2, A_3)$ is an object in $\text{Lact}(\underline{N})$ follows from condition (2) in the definition of ρ . That $[f_1]$ is a morphism in $\text{Sgp}(\underline{N})$ follows from the definition of γ . That $([f_1], [f_2])$ is a morphism in $\text{Lact}(\underline{N})$ follows from condition (1) in the definition of ρ . It is now clear that F is a functor.

Suppose $F(f) \circ F(g)$ is defined. Then $\text{dom } \hat{f} = (A_1, A_2, A_3)$ and $\text{cod } g = (A_1, A_2, A_3)$ for some (A_1, A_2, A_3) in $\underline{C}$. Thus, $f \circ g$ is defined so $F(f) \circ F(g) = F(f \circ g)$. Hence, the image of F is a subcategory of $\text{Lact}(\underline{N})$.

Suppose $F(\hat{f}) = F(\hat{g})$. Then $\hat{f} = (f_1, f_2)$ and $\hat{g} = (g_1, g_2)$ and $[f_1] = [g_1]$ and $[f_2] = [g_2]$ in $\underline{M}$. Thus, $[f_1] \hat{\rho} [g_1]$. Thus, as noted when proving $\otimes$ to be well-defined, there exist $h_1, \bar{h}_1, h_2, \bar{h}_2, \dots, h_n, \bar{h}_n$ so that $f_1 =$ or (1) or (2) $h_1 \times \bar{h}_1 =$ or (1) or (2) $h_2 \times \bar{h}_2 \cdots h_n \times \bar{h}_n =$ or (1) or (2) g_1 . Since $\text{dom } f_1$ has length one, (1) and (2) cannot hold, so we have $\hat{f}_1 = h_1$. As in the definition of γ , we obtain from h_1 and $\bar{h}_1$ the morphisms $H_1 = (H_1^1, \dots, H_1^1)$ and (from $\bar{h}_1$) the morphism $\bar{H}_1 = (\bar{H}_1^1, \dots, \bar{H}_1^1)$. Then since the length of $\text{dom } h_1 = 1$ and $\text{dom } h_1 = \text{dom } H_1 = \text{dom } \bar{H}_1 = \text{dom } \bar{h}_1$, we

have $H_1^{t1} \circ \dots \circ H_1^1 = \bar{H}_1^{\bar{t}1} \circ \dots \circ \bar{H}_1^1$. Since f_1 is a morphism in $\underline{C}_1$ and $f_1 = h_1$, then $H_1 = (1, h_1, 1)$ so $h_1 = H_1^{t1} \circ \dots \circ H_1^1$. Continuing in this manner, we obtain $f_1 = h_1 = H_1^{t1} \circ \dots \circ H_1^1 = \bar{H}_1^{\bar{t}1} \circ \dots \circ \bar{H}_1^1 = H_2^{t2} \circ \dots \circ H_2^1 = \dots = g_1$. Thus, $f_1 = g_1$. Similarly, $f_2 = g_2$. Hence, $f = g$. Thus, F is an embedding since F is one-to-one on morphisms. It is clear that each Γ_i -morphism in $\underline{C}$ maps to a Γ_i -morphism in $\text{Lact}(\underline{N})$, $i = 1, 2$. This completes the proof of 2.24.

CHAPTER III

A SUFFICIENT CONDITION FOR A WELL-FACTORED CATEGORY TO BE VARIETAL

In this chapter, we will be concerned with a type of varietal category introduced by Herrlich in [1] which generalizes the "varietal" categories of Lawvere and Linton.

(3.1). Remark. We recall the following definitions.

$(\underline{A}, U)$ is called a concrete category if and only if $U: \underline{A} \rightarrow \mathbf{Ens}$ and U is faithful. If p, q , and f are morphisms in $\underline{A}$, $\underline{A}$ any category, then (p, q) is called the congruence relation of f if and only if (p, q) is the pullback of f with itself. If $\underline{A}$ is any category, f is said to be an extremal epimorphism if and only if f is an epimorphism and $f = m \circ g$ is any factorization of f so that m is a monomorphism implies m is an isomorphism. A morphism f is said to be a regular epimorphism if and only if f is the coequalizer of some two morphisms. It is easy to see that if f is a regular epimorphism, then f is an extremal epimorphism.

(3.2). Definition (Herrlich). Let $(\underline{A}, U)$ be a concrete category. $(\underline{A}, U)$ is said to be an algebraic category if and only if it satisfies the following three conditions:

(A1) $\underline{A}$ has congruence relations and coequalizers.

(A2) U has a left adjoint.

(A3) U preserves and reflects regular epimorphisms.

$(\underline{A}, U)$ is said to be a varietal category if and only if it is algebraic and satisfies the following condition:

(V) U reflects congruence relations.

(3.3). Examples (Herrlich).

(1) The following are varietal categories: Ens , category of pointed sets, category of all groups, category of all semigroups, category of all monoids, $\underline{R}$, $\underline{R}^1$, ${}_R \underline{M}$ for any ring R , category of all unital R -modules where R is any ring with identity, category of all lattices, category of all Boolean algebras, category of all compact, Hausdorff spaces, category of all compact, Hausdorff groups, and Ab .

(2) The category of all torsion free abelian groups is an algebraic category which is not varietal.

(3.4). Remark. We now prove a sequence of propositions leading to our sufficient condition for a well-factored category to be algebraic or varietal. We state this sufficient condition in 3.5. The first few propositions will prove 3.6, which is an important part of the proof of 3.5. It may be helpful to the reader to verify the following propositions only for the category of all modules over all rings or for the category of monoids acting on pointed sets.

(3.5). Theorem. Let $\underline{M} = (\underline{M}, \otimes, \alpha)$ be a multiplicative category without unit such that $\underline{M}$ has finite coproducts. Also, assume that for each object M in $\underline{M}$ that $M \otimes _$ and $_ \otimes M$ preserve finite coproducts. Let $U_1: \text{Lact}(\underline{M}) \longrightarrow \text{Sgp}(\underline{M}) \times \underline{M}$ be the obvious forgetful functor. Assume the following to be true:

(1) If f is a regular epimorphism in $\text{Sgp}(\underline{N})$ and g is a regular epimorphism in $\underline{M}$, then $f \otimes g$ is an epimorphism in $\underline{M}$.

(2) $(\text{Sgp}(\underline{N}), U_2)$ and $(\underline{M}, U_3)$ are algebraic categories so that neither U_2 nor U_3 assumes the empty set as a value.

(3) $\text{Lact}(\underline{N})$ has congruence relations and coequalizers.

(4) If a and b are regular epimorphisms respectively in $\text{Sgp}(\underline{N})$ and in $\underline{M}$ and v is a monomorphism and the diagram below consisting of the solid arrows commutes, then there exists $z_{A,B}: A \otimes B \rightarrow B$ so that the entire diagram in $\underline{M}$ commutes:

$$\begin{array}{ccccc}
 A' \otimes B' & \xrightarrow{a \otimes b} & A \otimes B & \xrightarrow{u \otimes v} & A'' \otimes B'' \\
 \downarrow z_{A', B'} & & \downarrow z_{A, B} & & \downarrow z_{A'', B''} \\
 B' & \xrightarrow{b} & B & \xrightarrow{v} & B''
 \end{array}$$

Let $U = \Pi \circ (U_2 \times U_3) \circ U_1: \text{Lact}(\underline{N}) \rightarrow \text{Ens}$, where $\Pi: \text{Ens} \times \text{Ens} \rightarrow \text{Ens}$ is the product functor. Then $(\text{Lact}(\underline{N}), U)$ is an algebraic category. If, in addition, we assume that $(\text{Sgp}(\underline{N}), U_2)$ and $(\underline{M}, U_3)$ are varietal categories, then $(\text{Lact}(\underline{N}), U)$ is a varietal category.

(3.6). Proposition. Let $\underline{N}$ and U_1 be as in 3.5. Then U_1 has a left adjoint.

(3.7). Notation. We will let $\underline{N}$, U_1 , U_2 , U_3 , and Π be as in 3.5 for the rest of this chapter. Let u_E denote an injection into a coproduct involving E . Define $d_{A,B,C}: (A \otimes B) \amalg (A \otimes C) \rightarrow A \otimes (B \amalg C)$ to be the unique morphism d such that $d \circ u_{A \otimes B} = A \otimes u_B$ and $d \circ u_{A \otimes C} = A \otimes u_C$. Whenever possible, we will denote $d_{A,B,C}$ as d . Similarly, we define $\tilde{d}_{B,C,A}: (B \otimes A) \amalg (C \otimes A) \rightarrow (B \amalg C) \otimes A$. Whenever possible, we denote $\tilde{d}_{B,C,A}$

as $\bar{d}$. The assumption that $A \otimes _$ and $_ \otimes A$ preserve finite coproducts says that d and $\bar{d}$ are isomorphisms.

(3.8). Proposition. The transformations d and $\bar{d}$ are natural.

Proof. The proof of this proposition follows in a straightforward manner from the definition of a coproduct.

(3.9). Proposition. The following two diagrams commute:

(1)

$$\begin{array}{ccc}
 & A \otimes [(B \otimes C) \amalg (B \otimes D)] & \\
 \begin{array}{c} \nearrow d_{A \otimes B, C, B \otimes D} \\ \downarrow \alpha \amalg \alpha \end{array} & & \begin{array}{c} \nwarrow 1 \otimes d_{B, C, D} \\ \downarrow \alpha \end{array} \\
 [A \otimes (B \otimes C)] \amalg [A \otimes (B \otimes D)] & & A \otimes (B \otimes (C \amalg D)) \\
 \downarrow \alpha \amalg \alpha & & \downarrow \alpha \\
 [(A \otimes B) \otimes C] \amalg [(A \otimes B) \otimes D] & \xrightarrow{d_{A \otimes B, C, D}} & (A \otimes B) \otimes (C \amalg D)
 \end{array}$$

(2)

$$\begin{array}{ccc}
 & [(A \otimes C) \amalg (B \otimes C)] \otimes D & \\
 \begin{array}{c} \nearrow \bar{d}_{A \otimes B, B \otimes C, D} \\ \uparrow \alpha \amalg \alpha \end{array} & & \begin{array}{c} \nwarrow \bar{d}_{A, B, C \otimes 1} \\ \uparrow \alpha \end{array} \\
 [(A \otimes C) \otimes D] \amalg [(B \otimes C) \otimes D] & & ((A \amalg B) \otimes C) \otimes D \\
 \uparrow \alpha \amalg \alpha & & \uparrow \alpha \\
 [A \otimes (C \otimes D)] \amalg [B \otimes (C \otimes D)] & \xrightarrow{\bar{d}_{A, B, C \otimes D}} & (A \amalg B) \otimes (C \otimes D)
 \end{array}$$

Proof. To see that diagram (1) commutes, it suffices to show that (1) commutes when composed with the coproduct injection into the coproduct of $A \otimes (B \otimes C)$ and $A \otimes (B \otimes D)$.

We will prove it for $u_{A \otimes (B \otimes C)}$ only. We calculate the following equation. $d \circ (\alpha \amalg \alpha) \circ u_{A \otimes (B \otimes C)} = d \circ u_{(A \otimes B) \otimes C} \circ \alpha =$

$$[(1 \otimes 1) \otimes u_C] \circ \alpha = \alpha \circ (1 \otimes (1 \otimes u_C)) = \alpha \circ (1 \otimes (d \circ u_{B \otimes C})) = \alpha \circ (1 \otimes d) \circ (1 \otimes u_{B \otimes C})$$

$= \alpha \circ (1 \otimes d) \circ d \circ u_{A \otimes (B \otimes C)}$. Thus, by definition of coproduct, (1) commutes. The proof that diagram (2) commutes is analogous.

(3.10). Proposition. The following diagram commutes:

$$\begin{array}{ccc}
 [A \otimes (B \otimes D)] \amalg [A \otimes (C \otimes D)] & \xrightarrow{\alpha \amalg \alpha} & [(A \otimes B) \otimes D] \amalg [(A \otimes C) \otimes D] \\
 \downarrow d_{A, B \otimes D, C \otimes D} & & \downarrow \bar{d}_{A \otimes B, A \otimes C, D} \\
 A \otimes [(B \otimes D) \amalg (C \otimes D)] & & [(A \otimes B) \amalg (A \otimes C)] \otimes D \\
 \downarrow 1 \otimes \bar{d}_{B, C, D} & & \downarrow d_{A \otimes B, A \otimes C, D} \\
 A \otimes [(B \amalg C) \otimes D] & \xrightarrow{\alpha} & [A \otimes (B \amalg C)] \otimes D
 \end{array}$$

Proof. As in 3.9, it suffices to prove commutativity when composed with the injections into the coproduct of $A \otimes (B \otimes D)$ and $A \otimes (C \otimes D)$. Since the procedures are the same, we will consider only $u_{A \otimes (B \otimes D)}$. Then we have the following equalities: $\alpha \circ (1 \otimes \bar{d}) \circ d \circ u_{A \otimes (B \otimes D)} = \alpha \circ (1 \otimes \bar{d}) \circ (1 \otimes u_{B \otimes D}) = \alpha \circ (1 \otimes (\bar{d} \circ u_{B \otimes D})) = \alpha \circ (1 \otimes (u_B \otimes 1)) = ((1 \otimes u_B) \otimes 1) \circ \alpha = (d \otimes 1) \circ (u_{A \otimes B} \otimes 1) \circ \alpha = (d \otimes 1) \circ \bar{d} \circ u_{(A \otimes B) \otimes D} \circ \alpha = (d \otimes 1) \circ \bar{d} \circ (\alpha \amalg \alpha) \circ u_{A \otimes (B \otimes D)}$. Thus, the diagram commutes.

(3.11). Proposition. Let $R = (R, p)$ be an object in $\text{Sgp}(\underline{N})$. Let M be an object in $\underline{M}$. Define $F(R, M)$ to be $(R, M \amalg (R \otimes M), z)$ where z is given by $R \otimes (M \amalg (R \otimes M)) \xrightarrow{d} (R \otimes M) \amalg (R \otimes (R \otimes M)) \xrightarrow{1} (R \otimes M) \amalg ((R \otimes R) \otimes M) \xrightarrow{(1, p \otimes 1)} R \otimes M \xrightarrow{u_{R \otimes M}} M \amalg (R \otimes M)$, where $u_{R \otimes M}$ is the coproduct injection. Then $F(R, M)$ is an object in $\text{Lact}(\underline{N})$.

Proof. It suffices to show that the following diagram (0) commutes. Diagram (0) is redrawn in diagram (0') with the values of z indicated. Also, in (0'), diagram (0) is sub-

divided into eight subdiagrams, each of which is subsequently shown to commute. This constitutes the proof that (0) commutes.

$$\begin{array}{ccc}
 (0) & & \\
 (R \otimes R) \otimes (M \amalg (R \otimes M)) & \xrightarrow{p \otimes 1} & R \otimes (M \amalg (R \otimes M)) \\
 \uparrow \alpha & & \downarrow z \\
 R \otimes [R \otimes (M \amalg (R \otimes M))] & & \\
 \downarrow 1 \otimes z & & \\
 R \otimes (M \amalg (R \otimes M)) & \xrightarrow{z} & M \amalg (R \otimes M)
 \end{array}$$

Subdiagram 1 in (0') is redrawn in (1') and decomposed into diagrams 1.1, 1.2, 1.3, 1.4, and 1.5. Diagrams 1.1 and 1.5 commute by 3.10, 1.2 commutes since d is natural, and 1.3 and 1.4 commute by simple computations.

For subdiagram (2) to commute, it suffices to show $[(1, p) \otimes 1] \circ \tilde{d} = (1, p \otimes 1)$. It suffices here to show equality when we compose on both sides with the injections into the coproduct of $R \otimes M$ and $(R \otimes R) \otimes M$. Since the procedures are similar, we consider only $u_{R \otimes M}$. We have the following:
 $[(1, p) \otimes 1] \circ \tilde{d} \cdot u_{R \otimes M} = [(1, p) \otimes 1] \circ (u_R \otimes 1) = [(1, p) \cdot u_R] \otimes 1 = 1 \otimes 1 = 1 = (1, p \otimes 1) \cdot u_{R \otimes M}$. Thus, (2) commutes.

Diagram (3) commutes by the definition of d and diagram (4) commutes since d is natural.

Diagram (5) is redrawn as diagram (5') and is decomposed into subdiagrams 5.1, 5.2, and 5.3. Diagram 5.1 commutes by condition (d) in definition 2.1. Diagram 5.2 commutes since α is natural. Diagram 5.3 commutes since (R, p) is an object in $\mathcal{S}p(N)$.

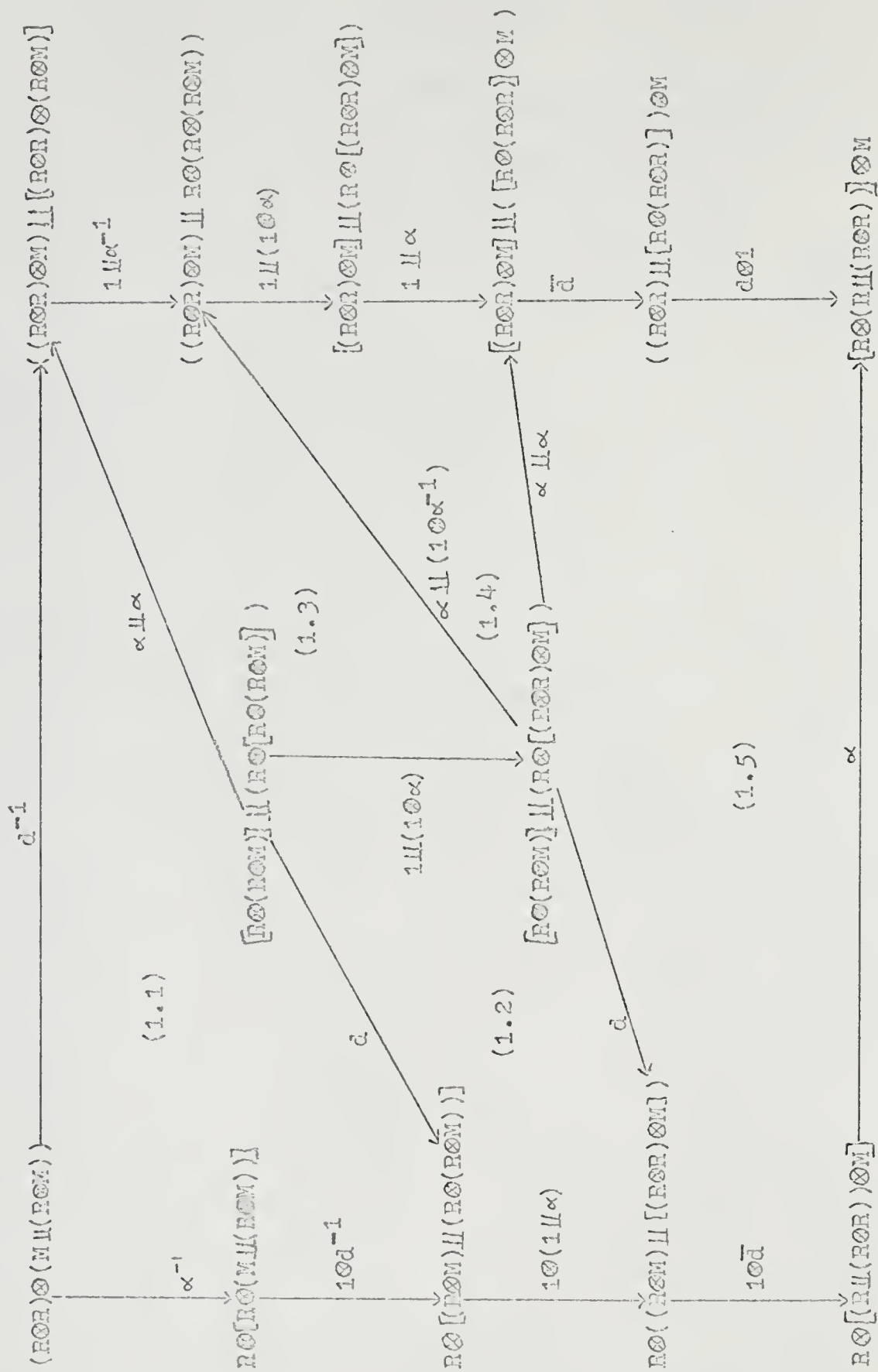


Diagram (1')

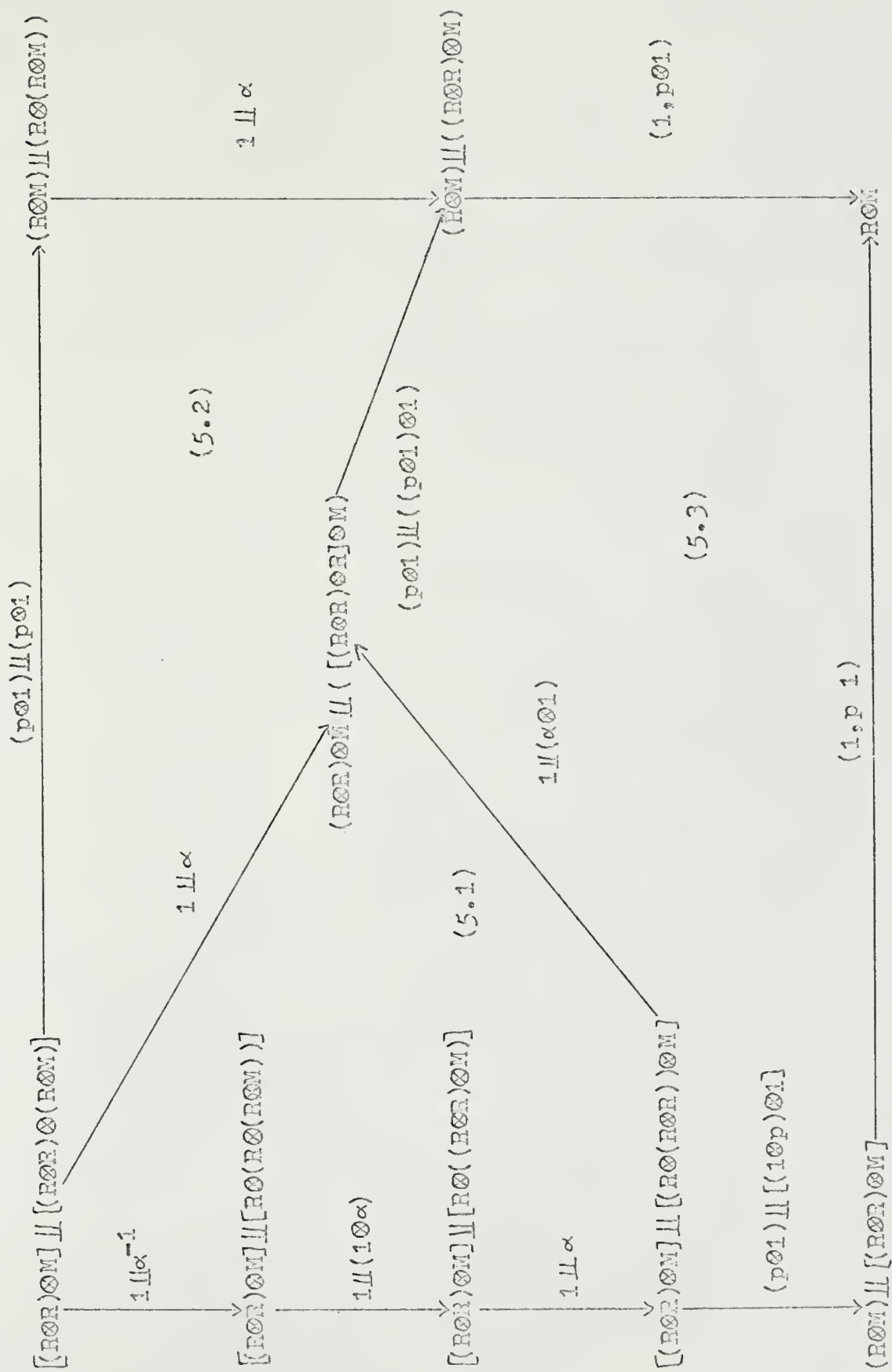


Diagram (5')

We note that diagram (6) commutes since $\tilde{d}$ is natural. For diagram (7) to commute, we need $[(1,p)\otimes 1]\circ\tilde{d} = (1,p\otimes 1)$. By the definition of coproduct, it suffices to prove the above equation with $u_{R\otimes M}$ or with $u_{(R\otimes R)\otimes M}$ composed with both sides. We obtain the equations: $[(1,p)\otimes 1]\circ\tilde{d}\circ u_{R\otimes M} = ((1,p)\otimes 1)\circ(u_R\otimes 1) = [(1,p)\circ u_R]\otimes 1 = 1\otimes 1 = 1 = (1,p\otimes 1)\circ u_{R\otimes M}$. The proof is just as easy for $u_{(R\otimes R)\otimes M}$ and is left for the reader. Thus, diagram (7) commutes.

To prove that diagram (8) commutes, we adjoin $u_{R\otimes R}\otimes 1$ and $u_{R\otimes(R\otimes R)}\otimes 1$ to the diagram to form respectively diagrams (8A) and (8B). It is clear by inspection that (8A) and (8B) commute. Thus, diagrams (1) through (8) commute so our original diagram (0) commutes. This completes the proof of 3.11.

(3.12). Proposition. Given that $(1,g):(R,M)\longrightarrow U_1(R,N,z)$ is a morphism in $\text{Sgp}(\underline{N})\times\underline{M}$, then there exists a unique morphism $(1,h)$ in $\text{Lact}(\underline{N})$ so that the following diagram commutes:

$$\begin{array}{ccc}
 (R,M) & \xrightarrow{(1,u_M)} & U_1 F(R,M) = (R,M \amalg (R\otimes M)) \\
 & \searrow (1,g) & \swarrow U_1(1,h) \\
 & U_1(R,N,z) & .
 \end{array}$$

Furthermore, $h = (g, z\circ(1\otimes g))$.

Proof. Commutativity of the above diagram is clear.

We must show $(1,(g,z\circ(1\otimes g)))$ is a morphism in $\text{Lact}(\underline{N})$.

It suffices to show the diagram below commutes:

$$\begin{array}{ccccccc}
 R\otimes(M \amalg (R\otimes M)) & \xrightarrow{d^{-1}} & (R\otimes M) \amalg [R\otimes(R\otimes M)] & \xrightarrow{(1 \amalg \alpha)} & (R\otimes M) \amalg [(R\otimes R)\otimes M] & \xrightarrow{(1,p\otimes 1)} & R\otimes M \\
 \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow u_{R\otimes M} \\
 R\otimes M & \xrightarrow{(g,z\circ(1\otimes g))} & R\otimes M & \xrightarrow{z} & R\otimes M & \xrightarrow{(g,z\circ(1\otimes g))} & R\otimes M
 \end{array}$$

As usual, we prove commutativity by adjoining first $u_{R \otimes M}$ and then $u_{R \otimes (R \otimes M)}$. We obtain the equations: $z \circ [1 \otimes (g, z \circ (1 \otimes g))] \circ d \circ u_{R \otimes M} = z \circ [1 \otimes (g, z \circ (1 \otimes g))] \circ (1 \otimes u_M) = z \circ [1 \otimes (g, z \circ (1 \otimes g)) \circ u_M] = z \circ (1 \otimes g) = [z \circ (1 \otimes g)] \circ 1 = [z \circ (1 \otimes g)] \circ (1, p \otimes 1) \circ u_{R \otimes M} \circ 1 = (g, z \circ (1 \otimes g)) \circ u_{R \otimes M} \circ (1, p \otimes 1) \circ (1 \parallel \alpha) \circ u_{R \otimes (R \otimes M)}$. (Note that we see from this and the required commutativity that h is unique.) $z \circ [1 \otimes (g, z \circ (1 \otimes g))] \circ d \circ u_{R \otimes (R \otimes M)} = z \circ [1 \otimes (g, z \circ (1 \otimes g))] \circ (1 \otimes u_{R \otimes M}) = z \circ (1 \otimes [(g, z \circ (1 \otimes g)) \circ u_{R \otimes M}]) = z \circ [1 \otimes (z \circ (1 \otimes g))] = z \circ (1 \otimes z) \circ [1 \otimes (1 \otimes g)]$. But, $(g, z \circ (1 \otimes g)) \circ u_{R \otimes M} \circ (1, p \otimes 1) \circ (1 \parallel \alpha) \circ u_{R \otimes (R \otimes M)} = (z \circ (1 \otimes g)) \circ (1, p \otimes 1) \circ u_{(R \otimes R) \otimes M} \circ \alpha = z \circ (1 \otimes g) \circ (p \otimes 1) \circ \alpha = z \circ (p \otimes g) \circ \alpha$. Thus, it suffices to prove the diagram below with subdiagrams (a) and (b) commutes:

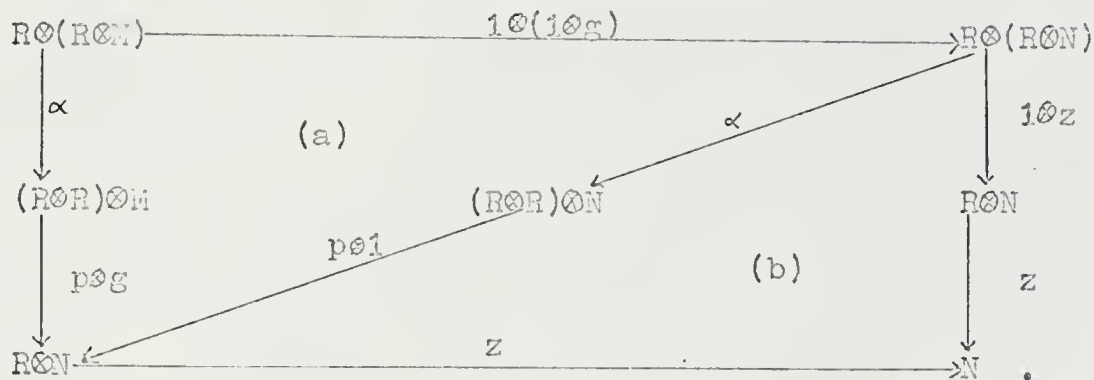


Diagram (a) is expanded below and clearly commutes:

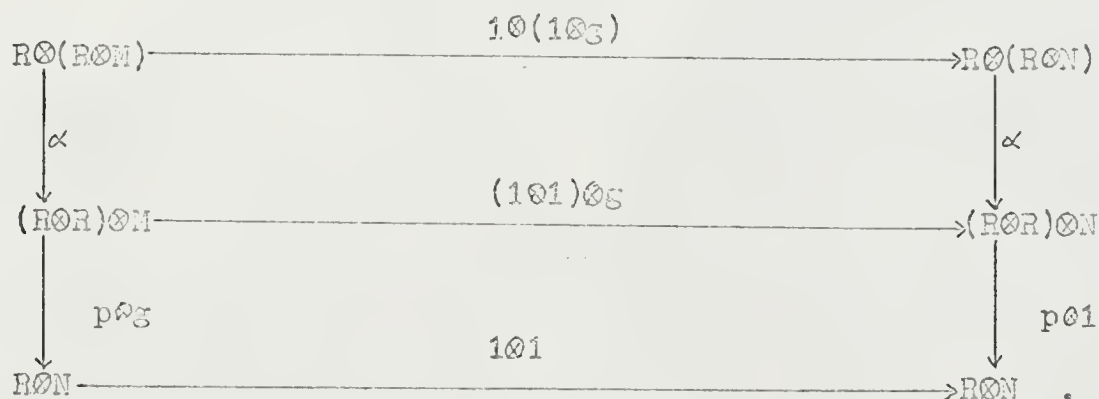


Diagram (b) commutes since (R, N, z) is an object in $\text{Lact}(\underline{N})$.

Since we have noted that h is unique, it is easy to see that

(1,h) is unique. This completes the proof of 3.12.

(3.13). Proposition. Let $\Gamma_1(R)$ denote the standard Γ_1 -subcategory of $\text{Lact}(\underline{N})$ determined by the semigroup R . Let $U_R = U_1|_{\Gamma_1(R)}$. Then $U_R: \Gamma_1(R) \rightarrow \{R\} \times \underline{M}$ has a left adjoint F . $F(R,M) = (R, M \amalg (R \otimes M), z)$ is the same as in 3.11. F is defined on morphisms by:

$$\begin{array}{ccc}
 (R,M) & \xrightarrow{(1, u_M)} & \times (R, M \amalg (R \otimes M), z_M) \\
 \downarrow (1,g) & & \downarrow F(1,g) = \\
 (R,N) & \xrightarrow{(1, u_N)} & \times (R, N \amalg (R \otimes N), z_N)
 \end{array}$$

$(1, (u_N \circ g, z_N \circ (1 \otimes (u_N \circ g))))$

Proof. This proposition follows immediately from 3.12.

(3.14). Remark. We now pause to prove some propositions of a different nature and which will be needed for the proof of 3.5.

(3.15). Proposition. Let $\underline{C}$ and $\underline{C}'$ be categories well-factored respectively by Γ_1, Γ_2 and Γ_1', Γ_2' . Let P be a class which indexes both Γ_1 and Γ_1' . Let $T: \underline{C}' \rightarrow \underline{C}$ be such that for each p in P , $T|_{G_p'}: G_p' \rightarrow G_p$. Let $T|_{G_p'} = T_p$. Assume that for each p in P , there exists $S_p: G_p \rightarrow G_p'$ so that S_p is left adjoint to T_p . Also, $T(C)$ is an object in G_p implies C is an object in G_p' . Finally, if $f: A \rightarrow T(B)$, then there exist unique f_1 and f_2 so that f_1 is a Γ_1 -morphism, f_2 is a Γ_2' -morphism, and $T(f_2) \circ f_1 = f$. Then there exists $S: \underline{C} \rightarrow \underline{C}'$ so that S is left adjoint to T and, for each p in P , $S|_{G_p} = S_p$.

Proof. Let A be an object in $\underline{C}$. Then there exists a unique G_p in Γ_1 so that A is an object in G_p . Define

$S(A) = S_p(A)$. Thus, $TS(A) = T_p S_p(A)$, so there exists $v: A \rightarrow TS(A)$, a morphism in G_p , so that if $f: A \rightarrow T_p(B)$ is in G_p , then there exists a unique $g: S(A) \rightarrow B$ in G_p^i so that $T(g) \circ v = f$.

Suppose $f: A \rightarrow T(B)$ is in $\underline{C}$ and that B is an arbitrary object of $\underline{C}$. By hypothesis, there exists unique f_1 , a G_p -morphism, and f_2 , a Γ_2^i -morphism, so that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{f_1} & T(C) \\
 & \searrow f & \swarrow T(f_2) \\
 & T(B) &
 \end{array}$$

Then $T(C)$ is in G_p so C is in G_p^i . Thus, $T(C) = T_p(C)$ and we have that there exists a unique g in G_p^i so that $T(g) \circ v = f_1$. Thus, the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{v} & TS(A) \\
 & \searrow f_1 & \swarrow T(g) \\
 & T(C) & \\
 & \searrow f & \swarrow T(f_2) \\
 & T(B) &
 \end{array}$$

We now prove that $f_2 \circ g$ is unique. Suppose $T(h) \circ v = f$. Then there exist h_1 , a Γ_1^i -morphism, and h_2 , a Γ_2^i -morphism, so that $h = h_2 \circ h_1$. But v is a G_p -morphism and so is $T(h_1)$, so $f = T(h_2) \circ (T(h_1) \circ v)$ implies $h_2 = f_2$ and $T(h_1) \circ v = f_1$. Then by uniqueness of g , $h_1 = g$ so $h = h_2 \circ h_1 = f_2 \circ g$. Thus, by the front adjunction theorem, S is the left adjoint of T . Also, S is defined on morphisms via the front ad-

junction diagram. Since S_p is the left adjoint of T_p ,
 $S|_{C_p} = S_p$, for each p in P .

(3.16). Behavior of S . The functor S is not necessarily well behaved on Γ_2 categories. In fact, it may be that $S(G)$, where G is any Γ_2 subcategory, is not contained in any Γ_2^i or Γ_1^i subcategory. We now give an example of this situation. Let $\underline{A}$ have objects A and B and non-identity morphisms $a:A \rightarrow B$ and $a^{-1}:B \rightarrow A$ with the indicated composition. Let $\underline{C} = \underline{C}' = \underline{A} \times \underline{A}$ have the standard well-factorization, where $\Gamma_1(\underline{C})$ denotes the Γ_1 subcategory of $\underline{A} \times \underline{A}$ determined by C , an object in $\underline{A}$. Let $S_A: \Gamma_1(A) \rightarrow \Gamma_1^i(A) = 1_{\Gamma_1(A)}$. Define $S_B: \Gamma_1(B) \rightarrow \Gamma_1^i(B)$ as follows: $S_B(B,A) = (B,B)$, $S_B(B,B) = (B,A)$, $S_B(1_{(B,A)}) = 1_{(B,B)}$, $S_B(1_{(B,B)}) = 1_{(B,A)}$, $S_B(1,p) = (1,p^{-1})$, and $S_B(1,p^{-1}) = (1,p)$. To see that S_B is a functor, we compute the following equations. $S_B((1,p) \circ (1,p^{-1})) = S_B(1,1) = 1_{(B,A)} = (1,p^{-1}) \circ (1,p) = S_B(1,p) \circ S_B(1,p^{-1})$. Similarly, $S_B((1,p^{-1}) \circ (1,p)) = S_B(1,p^{-1}) \circ S_B(1,p)$. Thus, S_B is a functor.

Let $T = 1_{\underline{A} \times \underline{A}}$. It is clear that all the hypotheses in 3.15 are satisfied, with the exception that S_B is the left adjoint of $T_B = 1_{\Gamma_1(B)}$. However, this is a consequence of the front adjunction theorem and of some easy calculations and the details will be left to the reader. Thus, by 3.15, there exists $S: \underline{C} \rightarrow \underline{C}'$ so that S is a left adjoint of T and $S|_{\Gamma_1(A)} = S_A$ and $S|_{\Gamma_1(B)} = S_B$. But $S(B,A) = S_B(B,A) = (B,B)$. Thus, (B,A) is an object in $\Gamma_2(A)$ but $S(B,A)$ is not an object in $\Gamma_2^i(A)$. In fact, $S(B,A)$ is an object in $\Gamma_2^i(B)$.

Thus, $S(\sqcap_2(A))$ is not contained in $\sqcap_2^0(A)$ or in $\sqcap_2^0(B)$.

(3.17). Proposition. There exists $F_1: \text{Sgp}(\underline{N}) \times \underline{M} \longrightarrow \text{Lact}(\underline{N})$ such that F_1 is a left adjoint of U_1 and $F_1|_{\{R\} \times \underline{M}}$ is the same as F in 3.13.

Proof. By 3.15, it suffices to show that for $(f,g):(R,M) \longrightarrow U_1(S,N,z)$, there exist unique $p_1:(R,M) \longrightarrow U_1(C)$, p_1 a $\sqcap_1$ -morphism, and $p_2:C \longrightarrow (S,N,z)$, p_2 a $\sqcap_2$ -morphism, so that $(f,g) = U_1(p_2) \circ p_1$. It suffices to show this since, by definition of U_1 and by 3.13, the remainder of the hypotheses in 3.15 are satisfied.

Let $C = (R,N,z \circ (f \otimes 1))$. By 2.22, C is an object in $\text{Lact}(\underline{N})$. Let $p_1 = (1,g)$. Clearly, p_1 is a $\sqcap_1$ -morphism in $\text{Sgp}(\underline{N}) \times \underline{M}$. Let $p_2 = (f,1)$. To see that p_2 is an act morphism, it suffices to note that the following diagram commutes:

$$\begin{array}{ccccc}
 R \otimes N & \xrightarrow{f \otimes 1} & S \otimes N & \xrightarrow{z} & N \\
 \downarrow f \otimes 1 & & & & \downarrow 1 \\
 S \otimes N & \xrightarrow{z} & & & N
 \end{array}$$

Thus, p_2 is a $\sqcap_2$ -morphism in $\text{Lact}(\underline{N})$. It is clear that the components of p_1 and of p_2 are uniquely determined. All that is left to show is the uniqueness of C . In particular, we must show that $z \circ [f \otimes 1]$ is unique. However, this follows from the fact that $(f,1)$ must be an act morphism and from, consequently, the commutativity of the above diagram. Thus, the factorization is unique. This completes the proof of this proposition.

(3.18). Remark. We see that 3.17 is just a restatement

of 3.6, so 3.6 is proved.

(3.19). Proposition. The product functor $\Pi: \text{Ens} \times \text{Ens} \longrightarrow \text{Ens}$ has a left adjoint F_4 , where $F_4(A) = (A, A)$ and $F_4(f) = (f, f)$.

Proof. We will prove this using the front adjunction theorem. For the remainder of this proof, we denote F_4 as F . Define $D_A: A \longrightarrow A \times A$ by $D_A(a) = (a, a)$, if A is non-empty. If A is the empty set, then let $D_A = A$. If $f: A \longrightarrow B$, then we have the following: $[(f \times f) \circ D_A](a) = (f \times f)(a, a) = (f(a), f(a)) = (D_B \circ f)(a)$. Thus, $D: 1_{\text{Ens}} \longrightarrow F$ is a natural transformation. Let $f: A \longrightarrow (C, E)$ ($= C \times E$). Let $p_1: C \times E \longrightarrow C$ and $p_2: C \times E \longrightarrow E$ be the usual projection functions. Then $(p_1 \circ f, p_2 \circ f): (A, A) \longrightarrow (C, E)$ is a morphism in $\text{Ens} \times \text{Ens}$. $\Pi(p_1 \circ f, p_2 \circ f) = (p_1 \circ f) \times (p_2 \circ f)$. $[(p_1 \circ f) \times (p_2 \circ f)] \circ D_A(a) = ((p_1 \circ f) \times (p_2 \circ f))(a, a) = (p_1(f(a)), p_2(f(a))) = f(a)$. Thus, $\Pi(p_1 \circ f, p_2 \circ f) \circ D_A = f$. Suppose $(u, v) \circ D_A = f$. Then if a is in A , $(u \times v)(a, a) = f(a)$, so $f(a) = (u(a), v(a))$. Thus, $(p_1 \circ f)(a) = u(a)$ and $(p_2 \circ f)(a) = v(a)$, so $p_1 \circ f = u$ and $p_2 \circ f = v$. Hence, $(p_1 \circ f, p_2 \circ f)$ is unique. By the front adjunction theorem, we then have that F is the left adjoint of Π .

(3.20). Proposition. The functor $U = \Pi \circ (U_2 \times U_3) \circ U_1$ of 3.5 has as left adjoint the functor $F = F_1 \circ (F_2 \times F_3) \circ F_4$ where F_1 , F_2 , F_3 , and F_4 are respectively the left adjoints of U_1 , U_2 , U_3 , and Π .

Proof. The left adjoints exist by 3.5 and by 3.20. and by 3.6. We refer only to the hypotheses of 3.5. The rest is straightforward and is left to the reader.

(3.21). Proposition. Let U_1 be as in 3.5. Suppose f is a regular epimorphism in $\text{Sgp}(\underline{N})$ and g is a regular epimorphism in $\underline{M}$ implies $f \circ g$ is an epimorphism in $\underline{M}$. Then U_1 reflects regular epimorphisms.

Proof. Let $(f, g): (R, M, z_M) \longrightarrow (S, N, z_N)$ be a morphism in $\text{Lact}(\underline{N})$. Suppose $U(f, g)$ is a regular epimorphism in $\text{Sgp}(\underline{N}) \times \underline{M}$, say $U(f, g)$ is $\text{coeq}((a, b), (a', b'))$ where $(a, b), (a', b'): (T, L) \longrightarrow (R, M)$. By proposition 3.12, $(1, (1, z_M \circ (1 \otimes 1))): F(R, M) \longrightarrow (R, M, z_M)$ is a morphism in $\text{Lact}(\underline{N})$. By 3.17, $F(a, b): F(T, L) \longrightarrow F(R, M)$ is also a morphism in $\text{Lact}(\underline{N})$. By 3.13 and by the proofs of 3.15 and 3.17, we have $F(a, b) = (a, h)$ where $h = (u_M \circ b, z \circ (a \otimes 1) \circ (1 \otimes (u_M \circ b)))$, where z is as described in 3.11. Thus, $(1, (1, z_M)) \circ F(a, b): F(T, L) \longrightarrow (R, M, z_M)$.
 $(1, (1, z_M)) \circ F(a, b) = (1, (1, z_M)) \circ (a, h) = (a, (1, z_M) \circ h)$. We wish to express $(1, z_M) \circ h$ in a simpler form. $(1, z_M) \circ h \circ u_L = (1, z_M) \circ u_M \circ b = 1 \circ b = b$. $(1, z_M) \circ h \circ u_{T \otimes L} = (1, z_M) \circ z \circ (a \otimes 1) \circ (1 \otimes (u_M \circ b)) = (1, z_M) \circ z \circ (a \otimes (u_M \circ b)) = (1, z_M) \circ z \circ (1 \otimes u_M) \circ (a \otimes b) = (1, z_M) \circ u_{R \otimes M} \circ (1, p \otimes 1) \circ (1 \otimes \alpha) \circ d^{-1} \cdot (1 \otimes u_M) \circ (a \otimes b) = z_M \circ (1, p \otimes 1) \circ (1 \otimes \alpha) \cdot u_{R \otimes M} \circ (a \otimes b) = z_M \circ (1, p \otimes 1) \circ u_{R \otimes M} \circ 1 \circ (a \otimes b) = z_M \circ 1 \circ (a \otimes b) = z_M \circ (a \otimes b)$. Thus, by the definition of coproduct, we have $(1, z_M) \circ h = (b, z_M \circ (a \otimes b))$. Hence, $(a, (b, z_M \circ (a \otimes b))): F(T, L) \longrightarrow (R, M, z_M)$ is a morphism in $\text{Lact}(\underline{N})$. Similarly, $(a', (b', z_M \circ (a' \otimes b'))): F(T, L) \longrightarrow (R, M, z_M)$ is a morphism in $\text{Lact}(\underline{N})$. In the above, p is the multiplication on R .
 $f \circ a = f \circ a'$. $g \circ (b, z_M \circ (a \otimes b)) \circ u_L = g \circ b = g \circ b' = g \circ (b', z_M \circ (a' \otimes b')) \circ u_L$. $g \circ (b, z_M \circ (a \otimes b)) \circ u_{T \otimes L} = g \circ z_M \circ (a \otimes b) = z_N \circ (f \otimes g) \circ (a \otimes b) = z_N \circ ((f \circ a) \otimes (g \circ b)) = z_N \circ ((f \circ a') \otimes (g \circ b')) = z_N \circ (f \otimes g) \circ (a' \otimes b') = g \circ z_M \circ (a' \otimes b') = g \circ (b', z_M \circ (a' \otimes b')) \circ u_{T \otimes L}$.

Thus, from the definition of coproduct, we have $g \circ (b, z_M \circ (a \otimes b)) = g \circ (b', z_M \circ (a' \otimes b'))$. This implies $(f, g) \circ (a, (b, z_M \circ (a \otimes b))) = (f, g) \circ (a', (b', z_M \circ (a' \otimes b')))$. Suppose $(u, v): (R, M, z_M) \longrightarrow (W, K, z_K)$ in $\text{Lact}(\underline{N})$ so that $(u, v) \circ (a, (b, z_M \circ (a \otimes b))) = (u, v) \circ (a', (b', z_M \circ (a' \otimes b')))$. Then, in the same manner as for (f, g) , we obtain $u \circ a = u \circ a'$ and $v \circ b = v \circ b'$. Then there exists a unique morphism (u^*, v^*) in $\text{Sgp}(\underline{N}) \times \underline{M}$ so that $(u, v) = (u^*, v^*) \circ (f, g)$. Since U_1 is faithful, uniqueness in $\text{Sgp}(\underline{N}) \times \underline{M}$ implies uniqueness in $\text{Lact}(\underline{N})$. Thus, it suffices to prove (u^*, v^*) is a morphism in $\text{Lact}(\underline{N})$ in order to prove that (f, g) is the coequalizer of $(a, (b, z_M \circ (a \otimes b)))$ and of the morphism $(a', (b', z_M \circ (a' \otimes b')))$ which proves that U_1 reflects regular epimorphisms. In the following diagram, portions (1), (2), and (3) and the outer rectangle commute:

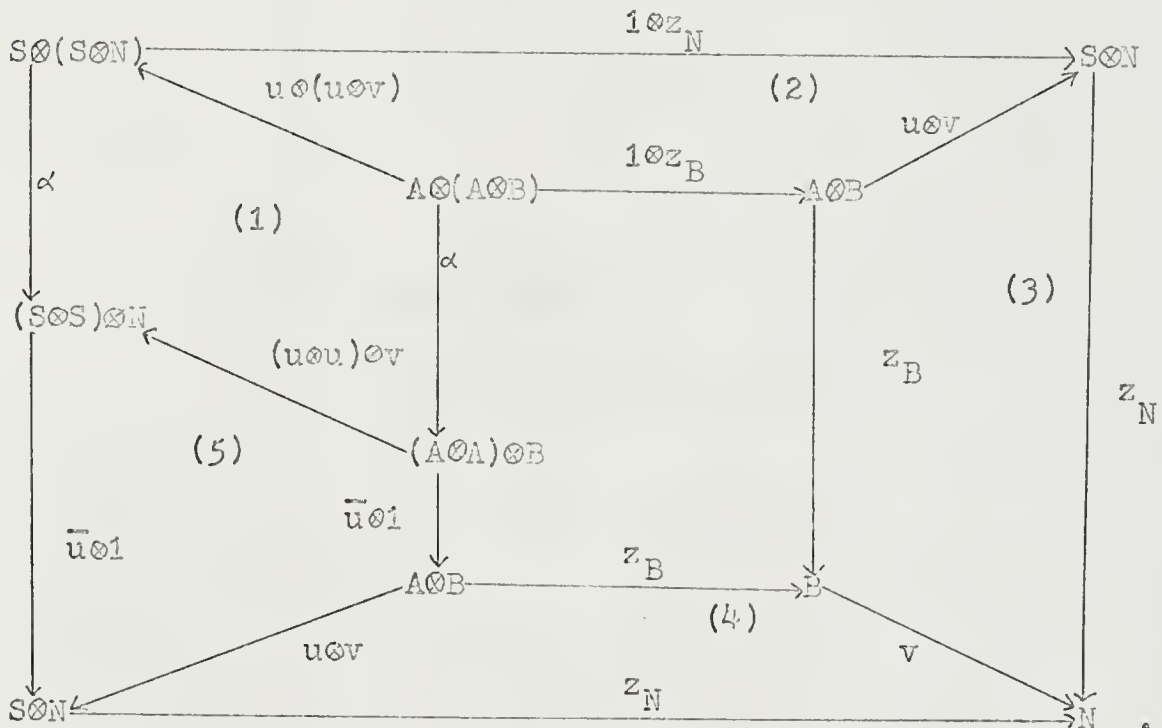
$$\begin{array}{ccccc}
 R \otimes M & \xrightarrow{z_M} & & & M \\
 \downarrow f \otimes g & \searrow u \circ v & \xrightarrow{(2)} & & \downarrow v \\
 & & W \otimes K & \xrightarrow{z_K} & K \\
 & \nearrow u^* \circ v^* & \xleftarrow{(3)} & & \nearrow v^* \\
 S \otimes N & \xrightarrow{z_N} & & & N
 \end{array}$$

Thus, we have $z_K \circ (u^* \circ v^*) \circ (f \otimes g) = v^* \circ z_N \circ (f \otimes g)$. Our hypothesis then implies that $f \otimes g$ is an epimorphism so $z_K \circ (u^* \circ v^*) = v^* \circ z_N$. Consequently, (u^*, v^*) is a morphism in $\text{Lact}(\underline{N})$. This completes the proof.

(3.22). Proposition. Suppose $\underline{M}$ and $\text{Sgp}(\underline{N})$ have regular epimorphism, monomorphism factorization of morphisms and that (4) in the hypotheses of 3.5 holds. Then U_1 of 3.5 preserves regular epimorphisms. In fact, U_1 maps extremal

epimorphisms onto regular epimorphisms.

Proof. Since a regular epimorphism is an extremal epimorphism, we may prove only that U_1 maps extremal epimorphisms onto regular epimorphisms. Let $(f, g): (R, M, z_M) \rightarrow (S, N, z_N)$ be an extremal epimorphism in $\text{Lact}(\underline{N})$. Then we have $v \circ b = g$ and $u \circ a = f$, where a and b are regular epimorphisms in respectively $\text{Sgp}(\underline{N})$ and $\underline{M}$ and u and v are monomorphisms respectively in $\text{Sgp}(\underline{N})$ and $\underline{M}$. Let $\text{dom } v = B$ and $\text{dom } u = A$. Then (f, g) is a morphism in $\text{Lact}(\underline{N})$ implies $g \circ z_M = z_N \circ (f \otimes g)$ so $v \circ b \circ z_M = z_N \circ (u \otimes v) \circ (a \otimes b)$. Then, by hypothesis, there exists $z_B: A \otimes B \rightarrow B$ so that $b \circ z_M = z_B \circ (a \otimes b)$ and $v \circ z_B = z_N \circ (u \otimes v)$. Let $\bar{u}$ be the semigroup multiplication on A . We have the following diagram:



Then (1) commutes by the naturality of α . Also, (2), (3), and (4) commute since $v \circ z_B = z_N \circ (u \otimes v)$. Finally, (5) commutes since u is a morphism in $\text{Sgp}(\underline{N})$. Then we have that

$v \circ z_B \circ (\bar{u} \otimes 1) \circ \alpha = v \circ z_B \circ (1 \otimes z_B)$. Since v is a monomorphism, this implies $z_B \circ (\bar{u} \otimes 1) \circ \alpha = z_B \circ (1 \otimes z_B)$. Thus, (A, B, z_B) is an object in $\text{Lact}(\underline{N})$. Then we have (a, b) is a morphism in $\text{Lact}(\underline{N})$. Also, (u, v) is a monomorphism in $\text{Lact}(\underline{N})$. But since (f, g) is an extremal epimorphism, $(f, g) = (u, v) \circ (a, b)$ implies (u, v) is an isomorphism in $\text{Lact}(\underline{N})$ so $U_1(u, v)$ is an isomorphism in $\text{Sgp}(\underline{N}) \times \underline{M}$. But $U_1(f, g) = U_1(u, v) \circ U_1(a, b)$ and $U_1(a, b)$ is a regular epimorphism in $\text{Sgp}(\underline{N}) \times \underline{M}$ since a and b are regular epimorphisms respectively in $\text{Sgp}(\underline{N})$ and $\underline{M}$. Consequently, $U_1(f, g)$ is a regular epimorphism in $\text{Sgp}(\underline{N}) \times \underline{M}$ so U_1 preserves regular epimorphisms.

(3.23). Proposition. U_1 of 3.5 reflects congruence relations.

Proof. Suppose the following is a diagram in $\text{Lact}(\underline{N})$:

$$\begin{array}{ccc}
 (P, Q, z_Q) & \xrightarrow{(p_2, q_2)} & (R, M, z_M) \\
 \downarrow (p_2, q_2) & & \downarrow (f, g) \\
 (R, M, z_M) & \xrightarrow{(f, g)} & (S, N, z_N)
 \end{array}$$

Denote $U_1(P, Q, z_Q)$ as (P, Q) and $U_1(p_1, q_1)$ as (p_1, q_1) , and so forth. Suppose the following is a pullback square in $\text{Sgp}(\underline{N}) \times \underline{M}$:

$$\begin{array}{ccc}
 (P, Q) & \xrightarrow{(p_1, q_1)} & (R, M) \\
 \downarrow (p_2, q_2) & & \downarrow (f, g) \\
 (R, M) & \xrightarrow{(f, g)} & (S, N)
 \end{array}$$

Suppose that the following diagram commutes in $\text{Lact}(\underline{N})$:

$$\begin{array}{ccc}
 (P', Q', z_{Q'}) & \xrightarrow{(p_1', q_2')} & (R, M, z_M) \\
 \downarrow (p_2', q_2') & & \downarrow (f, g) \\
 (R, M, z_M) & \xrightarrow{(f, g)} & (S, N, z_N)
 \end{array}$$

Then we also have $(f, g) \circ (p_1', q_1') = (f, g) \circ (p_2', q_2')$ in $\text{Sgp}(\underline{N}) \times \underline{M}$. Hence, there exists a unique $(p, q): (P', Q') \longrightarrow (P, Q)$ such that, in $\text{Sgp}(\underline{N}) \times \underline{M}$, $(p_i, q_i) \circ (p, q) = (p_i', q_i')$, for $i = 1, 2$. Since U_1 is faithful, uniqueness in $\text{Sgp}(\underline{N}) \times \underline{M}$ implies uniqueness in $\text{Lact}(\underline{N})$. Thus, it suffices to show that $(p, q): (P', Q', z_{Q'}) \longrightarrow (P, Q, z_Q)$ is a morphism in $\text{Lact}(\underline{N})$. Since (q_1, q_2) is the pullback in $\underline{M}$ of g with itself and since $g \circ q_1' \circ z_{Q'} = g \circ q_2' \circ z_{Q'}$, we have that there exists a unique $r: P' \otimes Q' \longrightarrow Q$ so that $q_i \circ r = q_i' \circ z_{Q'}$, for $i = 1, 2$. However, $q_1 \circ q \circ z_{Q'} = q_1' \circ z_{Q'}$, $i = 1, 2$. Also, $q_1 \circ z_Q \circ (p \otimes q) = z_M \circ (p_1 \otimes q_1) \circ (p \otimes q) = z_M \circ (p_1' \otimes q_1') = q_1' \circ z_{Q'}$. Thus, $q \circ z_{Q'} = r = z_Q \circ (p \otimes q)$. Consequently, $(p, q): (P', Q', z_{Q'}) \longrightarrow (P, Q, z_Q)$ is a morphism in $\text{Lact}(\underline{N})$. Thus, U_1 reflects congruence relations.

(3.24). Proposition. Let $(\underline{A}, U)$ and $(\underline{B}, V)$ be algebraic categories so that neither U nor V assumes the empty set as a value. Then $(\underline{A} \times \underline{B}, \Pi^*(U \times V))$ is an algebraic category, where Π is as in 3.19.

Proof. If F and G are respectively the left adjoints of U and V and if H denotes the left adjoint of Π (see 3.19), then it is easy to see that $(F \times G) \circ H$ is the left adjoint of $\Pi^*(U \times V)$. The remaining properties are easily verified.

(3.25). Proposition. Proposition 3.24 holds with "algebraic" replaced by "varietal."

Proof. The proof is straightforward and is left to the reader.

(3.26). Proof of 3.5. This is immediate from propositions 3.6 and 3.21 through 3.25 and from the fact that since $\text{Sgp}(\underline{N})$ and $\underline{M}$ are algebraic, they have regular epimorphism, monomorphism factorization of morphisms (see [1]).

(3.27). Corollary to 3.5. $\underline{R}\underline{M}$, the category of all modules over all rings, is a varietal category.

Proof. Let $\underline{N} = (\text{Ab}, \otimes, \alpha)$ where $\otimes$ denotes the tensor product over the integers and α is the usual associativity transformation. Then it is well known that $\underline{N}$ is a multiplicative category without unit. Also, it is well known that for any abelian group E , $_ \otimes E$ and $E \otimes _$ preserve finite (in fact, arbitrary) coproducts. (For details about the tensor product, the reader may see [6].)

It is easy to see that $\text{Sgp}(\underline{N})$ is isomorphic to $\underline{R}$ and that $\text{Lact}(\underline{N})$ is isomorphic to $\underline{R}\underline{M}$. Since isomorphisms preserve all the properties concerned, we will check in $\underline{R}\underline{M}$ the hypotheses to 3.5.

(1) Let f be a regular epimorphism in $\underline{R}$ and g be a regular epimorphism in Ab . Then f and g are onto functions so $f \otimes g$ is an onto function. Hence, $f \otimes g$ is an epimorphism. (For background material on $\underline{R}$, see [7].)

(2) It is well known that Ab and $\underline{R}$ with their forgetful functors are varietal categories and that their forgetful functors never assume the empty set as a value.

(3) Let $(f, g): (R, K, \cdot) \longrightarrow (S, N, *)$ be a morphism in $\underline{R}\underline{M}$. It is easy to show that the following is a congruence relation

where p_i and q_i are the natural projections, $i = 1, 2$, and where we use the induced module multiplication on the group $\{(m, m') \mid g(m) = g(m')\}$:

$$\begin{array}{ccc}
 \{(r, r') \mid f(r) = f(r')\}, \{(m, m') \mid g(m) = g(m')\}, \cdot & \xrightarrow{(p_1, q_1)} & (R, M, \cdot) \\
 \downarrow (p_2, q_2) & & \downarrow (f, g) \\
 (R, M, \cdot) & \xrightarrow{(f, g)} & (S, N, *)
 \end{array}$$

The details are left to the reader.

Now we construct coequalizers. Let (f, g) and (f', g') taking $(R, M, \cdot)$ into $(S, N, *)$ be two morphisms in $\underline{R}\underline{M}$. Let $N' = \{g(m) - g'(m) \mid m \in M\}$. Let I be the ideal of S generated by $\{f(r) - f'(r) \mid r \in R\}$. It is easy to see that $p: S \rightarrow S/I$ is the coequalizer of f and f' in $\underline{R}$, where p is the natural projection. There is an induced module multiplication for $(S/I, N/(N' + IN + SN'))$. Let $q: N \rightarrow N/(N' + IN + SN')$ be the natural map. By definition of the induced multiplication, (p, q) is a morphism in $\underline{R}\underline{M}$. It is clear that $(p, q) \circ (f, g) = (p, q) \circ (f', g')$. Suppose $(k, t) \circ (f, g) = (k, t) \circ (f', g')$, where $(k, t): (S, N, *) \rightarrow (T, L, \cdot)$. Since $p = \text{coeq}(f, f')$, there exists a unique ring morphism k^* so that $k^* \circ p = k$. Define $t^*(q(n)) = t(n)$. To see that t^* is well-defined, suppose x is in N' , IN , or SN' . If x is in N' , then $x = g(m) - g'(m)$, for some m , so $t(x) = t(g(m) - g'(m)) = t(g(m)) - t(g'(m)) = 0$. For each r in R , $t((f(r) - f'(r))n) = k(f(r) - f'(r))t(n) = 0 \cdot t(n) = 0$. Since I is generated by $\{f(r) - f'(r) \mid r \in R\}$, $t(x) = 0$, for each x in IN . $t(s(g(m) - g(m')))) = k(s)t(g(m) - g(m')) = k(s) \cdot 0 = 0$. Thus, for each x in SN' ,

$t(x) = 0$. Hence, t^* is well-defined. It is clear that t^* is a morphism in Ab and it is left to the reader to verify that (k^*, t^*) is a morphism $\underline{R}\underline{M}$. Since p and q are onto, (k^*, t^*) is unique. Thus, (p, q) is the coequalizer of (f, g) and (f^*, g^*) .

(4) Suppose a and b are regular epimorphisms respectively in $\underline{R}$ and Ab . Also, assume that v is a monomorphism in Ab . Then a , b , and $a \otimes b$ are onto and v is one-to-one. Suppose the following diagram in Ab commutes:

$$\begin{array}{ccccc}
 A' \otimes B' & \xrightarrow{a \otimes b} & A \otimes B & \xrightarrow{u \otimes v} & A'' \otimes B'' \\
 \downarrow z_{B'} & & & & \downarrow z_{B''} \\
 B' & \xrightarrow{b} & B & \xrightarrow{v} & B''
 \end{array}$$

Define $z_B: A \otimes B \rightarrow B$ as follows. For x in $A \otimes B$, $a \otimes b$ is onto implies that there exists x' in $A' \otimes B'$ so that $(a \otimes b)(x') = x$. Define $z_B(x) = (b \cdot z_{B'})(x')$. It is left to the reader to show that z_B is well-defined, is a morphism in Ab , and that the required commutativity holds.

Thus, all the hypotheses of 3.5 are satisfied. Hence, by 3.5, $\underline{R}\underline{M}$ is a varietal category.

(3.28). Generalizations and Further Results. We have obtained a generalization of 3.5 which allows us to deduce that certain subcategories of $\text{Lact}(\underline{N})$ are varietal categories. Due to time pressures, we are unable to include the details. However, we would like to list a number of well known categories which are varietal. The proofs of these assertions are along the same lines as the proof that $\underline{R}\underline{M}$ is varietal.

(3.29). Varietal Categories. The following categories

are varietal:

(a) the category of all not necessarily unital modules over all rings having an identity, where ring morphisms preserve identities and morphisms of the category are analogous to morphisms in $\underline{R}\underline{M}$,

(b) the category of all monoids acting on pointed sets, not necessarily in a unital fashion but so that if $(M, (X, x), \cdot)$ is an object in the category, then $1 \cdot x = x$, where morphisms in the category are analogous to morphisms in $\underline{R}\underline{M}$, and

(c) the category of all compact, Hausdorff monoids acting on compact, Hausdorff spaces analogously as in (b). It is our hope to publish at a later date a complete proof of 3.29.

(3.30). Categories Which Are Almost Varietal. Let $\underline{C}$ be either the category of all compact acts (see 1.17) or the category of all semigroups acting on sets. Let $U: \underline{C} \rightarrow \mathbf{Ens} \times \mathbf{Ens}$ be the forgetful functor. Then U satisfies properties A2, A3, and V in definition 3.2. Furthermore, $\underline{C}$ has congruence relations and coequalizers.

Proof. U has a left adjoint by 3.6. We hope that the remainder of this proof will be published at a later date. Due to time pressures, it is not given now.

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BIOGRAPHICAL SKETCH

Stephen Jackson Maxwell was born September 21, 1945, in Plant City, Florida. In June, 1963, he graduated from Plant City Senior High School. In June, 1966, he received the Bachelor of Arts in mathematics from the University of South Florida. In August, 1967, he received the Master of Arts in mathematics from the University of South Florida. From September, 1967, until the present time, he has studied at the University of Florida toward the degree of Doctor of Philosophy. He has held throughout his stay at the University of Florida a NASA Traineeship. During the summer of 1969, he attended an NSF Advanced Science Seminar in Category Theory at Bowdoin College in Maine. He is a member of the American Mathematical Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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